MATH10242 Sequences and Series

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[1]Based on the notes of Profs A. J. Wilkie, J. T. Stafford and M. Prest

Contents

0.1 Introduction

Maybe you can see that

$$1+\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\cdots+\frac{1}{2^n}+\cdots=2$$

and even that

$$1+\frac{1}{3}+\frac{1}{9}+\frac{1}{27}+\cdots+\frac{1}{3^n}+\cdots=\frac{3}{2}.$$

But what exactly do these formulas mean? What does it mean to add infinitely many numbers together? Is that even meaningful?

You might recognize the numbers above as being terms of geometric progressions and know the relevant general formula, for $|x|<1$,

$$1+x+x^2+x^3+\cdots+x^n+\cdots=\frac{1}{1-x}.$$

But how do we prove this? And what if $|x|\geq 1$?

In this course we shall answer these, and related, questions. In particular, we shall give a rigorous definition of what it means to add up infinitely many numbers and then we shall find rules and procedures for finding the sum in a wide range of particular cases.

Here are some more, rather remarkable, such formulas:

$$1+\frac{1}{4}+\frac{1}{9}+\frac{1}{16}+\cdots+\frac{1}{n^2}+\cdots=\frac{\pi^2}{6},$$

$$1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+\cdots+\frac{(-1)^{n+1}}{n}+\cdots=\ln 2,$$

$$1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\cdots+\frac{1}{n}+\cdots=\infty.$$

We shall prove the second and third of these formulas in this course unit, but the first one is too difficult and will be done in your lectures on real and complex analysis in the second year. Here, "real analysis" means the study of functions from real numbers to real numbers, from the point of view of developing a rigorous foundation for calculus (differentiation and integration) and for other infinite processes.[2] The study of sequences

[2]and "complex analysis" refers to the complex numbers, not (necessarily) complexity in the sense of "complicated"!

and series is the first step in this programme.

This also means there are two contrasting aspects to this course. On the one hand we will develop the machinery to produce formulas like the ones above. On the other hand it is also crucial to understand the theory that lies behind that machinery. This rigorous approach forms the second aspect of the course—and is in turn the first step in providing a solid foundation for real analysis.

Chapter 1

Before We Begin

1.1 Some Reminders about Mathematical Notation

1.1.1 Special sets

We use the following notation throughout the course.

$\mathbb{R}$ - the set of real numbers;

$\mathbb{R}^+$ - the set of strictly positive real numbers, i.e. $\mathbb{R}^+ = \{x \in \mathbb{R} : x > 0\}$;

$\mathbb{Q}$ - the set of rational numbers;

$\mathbb{Z}$ - the set of integers (positive, negative and 0);

$\mathbb{N}$ (or $\mathbb{Z}^+$) - the set of natural numbers, or **positive** integers $\{x \in \mathbb{Z} : x > 0\}$. (In this course, we do not count 0 as a natural number. We can use some other notation like $\mathbb{Z}^{\geq 0}$ for the set of integers greater than or equal to 0.)

$\emptyset$ - the empty set.

1.1.2 Set theory notation

The expression "$x \in X$" means x is an **element** (or **member**) of the set X. For sets A, B, we write $A \subseteq B$ to mean that A is a **subset** of B (i.e. every element of A is also an element of B). Thus $\emptyset \subseteq \mathbb{N} \subseteq \mathbb{Z} \subseteq \mathbb{Q} \subseteq \mathbb{R}$.

Standard **intervals** in $\mathbb{R}$: if $a, b \in \mathbb{R}$ with $a \leq b$, then

- $(a, b) = \{x \in \mathbb{R} : a < x < b\}$;
- $[a, b) = \{x \in \mathbb{R} : a \leq x < b\}$;
- $(a, b] = \{x \in \mathbb{R} : a < x \leq b\}$;
- $[a, b] = \{x \in \mathbb{R} : a \leq x \leq b\}$;

- $(a,\infty) = \{x \in \mathbb{R} : a < x\}$;
- $[a,\infty) = \{x \in \mathbb{R} : a \leq x\}$;
- $(-\infty,b) = \{x \in \mathbb{R} : x < b\}$;
- $(-\infty,b] = \{x \in \mathbb{R} : x \leq b\}$;
- $(-\infty,\infty) = \mathbb{R}$.

1.1.3 Logical notation

The expression "$\forall x$ ..." means "for all x ..." and "$\exists x$ ..." means "there exists at least one x such that ...". These are usually used in the context "$\forall x \in A$ ..." meaning "for all elements x of the set A ...", and "$\exists x \in A$ ..." meaning "there exists at least one element x in the set A such that ...".

Thus, for example, "$\forall x \in \mathbb{R}\ x > 1$" means "for all real numbers x, x is greater than 1" (which happens to be false) and "$\exists x \in \mathbb{R}\ x > 1$" means "there exists a real number x such that x is greater than 1" (which happens to be true).

1.1.4 Greek letters

The two most commonly used Greek letters in this course are δ (delta) and ε (epsilon). They are reserved exclusively for (usually small) positive real numbers.

Others are α (alpha), β (beta), γ (gamma), λ (lambda), θ (theta-usually an angle), η (eta) and Σ (capital sigma - the summation sign which will be used when we come to study series in Part II).

1.1.5 Where we're headed and some things we'll see on the way

In Part I we aim to understand the behaviour of infinite sequences of real numbers, meaning what happens to the terms as we go further and further on in the sequence. Do the terms all gradually get as close as we like to a limiting value (then the sequence is said to converge to that value) or not? The "conceptual" aim here is to really understand what this means. To do that, we have to be precise and avoid some plausible but misleading ideas. It's worthwhile trying to develop, and refine, your own "pictures" of what's going on. We also have to understand the precise definition well enough to be able to use it when we calculate examples, though we will gradually build up a stock of general results (the "Algebra of Limits"), general techniques and particular cases, so that we don't have to think so hard when faced with the next example.

Part II is about "infinite sums" of real numbers: how we can make a sensible definition of that vague idea and then how we can calculate the value of an infinite sum - if it exists.

We also need to be able to tell whether a particular "infinite sum" does or doesn't make sense/exist. Sequences appear here in two ways: first as the *sequence* of numbers to be "added up" (and the order of adding up does matter, as we shall see); second as a crucial ingredient in the actual definition of an "infinite sum" ("infinite series" is the official term). What we actually do is add up just the first n terms of such an infinite series - call this value the n-th partial sum - and then see what happens to this sequence (note) of partial sums as n gets bigger and bigger. If that sequence of partial sums converges to a limit then that limit is what we define to be the sum of the infinite series. Hopefully that makes sense to you and seems like it should be the right definition to use. Anyway, it works and, again, we have the conceptual aspect to get to grips with as well as various techniques that we can (try to) use in computations of examples.

Here are some of the things we prove about our concept of limit: a sequence can have *at most* one limit; if a sequence is increasing but never gets beyond a certain value, then it has a limit; if a sequence is squeezed between two other sequences which have the same limit l, then it has limit l. These properties help clarify the concept and are frequently used in arguments and calculations. We also show arithmetic properties like: if we have two sequences, each with a limit, and produce a new sequence by adding corresponding terms, then this new sequence has a limit which is, as you might expect, the sum of the limits of the two sequences we started with.

Then we turn to methods of calculating limits. We compare standard functions (polynomials, logs, exponentials, factorials, ...) according to how quickly they grow, but according to a very coarse measure - their "order" of growth, rather than rate of growth (i.e. derivative). That lets us see which functions in a complicated expression for the n-th term of a sequence are most important in calculating the limit of the sequence. There will be lots of examples, so that you can gain some facility in computing limits, and there are various helpful results, L'Hôpital's Rule being particularly useful.

While the properties of sequences are, at least once you've absorbed the concept, quite natural, infinite series hold quite a few surprises and really illustrate the need to be careful about definitions in mathematics (many mathematicians made errors by mishandling infinite series, especially before the concepts were properly worked out in the 1800s). Given an infinite series, there are two questions: does it have a sum? (then we say that it "converges", meaning that the sequence of partial sums has a limit - the value of that infinite sum) and, if so, what is the value of the sum? There are a few series (e.g. a geometric series with ratio < 1) where we can quite easily compute the value but, in general this is hard. It is considerably easier to determine *whether* a series has a sum or not by comparing it with a series we already know about. Indeed, the main test for convergence that we will use, the Ratio Test, is basically comparison with a geometric series.

Many infinite series that turn up naturally are "alternating", meaning that the terms are alternately positive and negative. So, in contrast with the corresponding series where all the terms are made positive, there's more chance of an alternating sequence converging, because the terms partly cancel each other out. Indeed, remarkably, it's certain provided the absolute values of the individual terms shrink monotonically to 0!

We'll finish with power series: infinite series where each term involves some variable x - you could think of these as "infinite polynomials in x". Whether or not a power series converges (i.e. has a sum), depends very much on the value of x, convergence being more likely for smaller values of x. In fact, the picture is as good as it could be: there's a certain "radius of convergence" R (which could be 0 or ∞ or anything in between, depending on the series) such that, within that radius we have convergence, outside we have divergence, and on the boundary ($x = \pm R$) it could go either way for each boundary point (so we have to do some more work there).

1.1.6 Basic properties of the real numbers

It will be assumed that you are familiar with the elementary properties of $\mathbb{N}$, $\mathbb{Z}$ and $\mathbb{Q}$ that were covered last semester in MATH10101/10111. These include, in particular, the basic facts about the arithmetic of the integers and a familiarity with the Principle of Mathematical Induction. One may then proceed to construct the set $\mathbb{R}$ of real numbers. There are many ways of doing this which, remarkably, all turn out to be equivalent in a sense that can be made mathematically precise. One method with which you should be familiar is to use infinite decimal expansions as described in Section 13.3 of [PJE].

Here we extract some of the basic arithmetic and order properties of the real numbers.

First, just as for the set of rational numbers, $\mathbb{R}$ is a **field**. That means that it satisfies the following conditions.

(A0) $\forall a, b \in \mathbb{R}$ one can form the sum $a + b$ and the product $a \cdot b$ (also written as just ab). We have that $a + b \in \mathbb{R}$ and $a \cdot b \in \mathbb{R}$; (*existence of two binary operations.*)

Properties of addition.

(A1) $\forall a, b, c \in \mathbb{R}$, $a + (b + c) = (a + b) + c$ (*associativity* of $+$);

(A2) $\forall a, b \in \mathbb{R}$, $a + b = b + a$ (*commutativity* of $+$);

(A3) $\exists 0 \in \mathbb{R}, \forall a \in \mathbb{R},\ a + 0 = 0 + a = a$ (*additive identity*);

(A4) $\forall a \in \mathbb{R}$, there exists an element in $\mathbb{R}$ (denoted $-a$) such that $a + (-a) = 0 = (-a) + a$; (*additive inverse*);

Properties of multiplication.

(A5) $\forall a, b, c \in \mathbb{R},\ a \cdot (b \cdot c) = (a \cdot b) \cdot c$ (*associativity* of $\cdot$);

(A6) $\forall a, b \in \mathbb{R},\ a \cdot b = b \cdot a$ (*commutativity* of $\cdot$);

(A7) $\exists 1 \in \mathbb{R}, \forall a \in \mathbb{R},\ a \cdot 1 = 1 \cdot a = a$ (*multiplicative identity*);

(A8) $\forall a \in \mathbb{R}$, if $a \neq 0$ then there exists an element in $\mathbb{R}$ (denoted a^{-1} or $1/a$) such that $a \cdot a^{-1} = 1 = a^{-1} \cdot a$ (*multiplicative inverse*);

Combining the two operations.

(A9) $\forall a, b, c \in \mathbb{R},\ a \cdot (b + c) = a \cdot b + a \cdot c$ (the *distributive* law).

These **axioms** (A0)-(A9) list the basic arithmetic/algebraic properties that hold in $\mathbb{R}$ and from which all the other such properties may be deduced.

Importantly,

Example A The identities and inverses, both additive and multiplicative, are unique.

Solution If 0 and $0'$ are two additive identities then

$$\begin{aligned} 0 &= 0 + 0' \quad \text{since } 0' \text{ is an additive identity} \\ &= 0' \quad \text{since } 0 \text{ is an additive identity.} \end{aligned}$$

Thus $0 = 0'$, the identity is unique.

If a has two additive inverses, b and c then $0 = a + b$ since b is the additive inverse of a. Adding c to both sides gives

$$c + 0 = c + (a + b).$$

On the left use that 0 is the additive identity and on the right use associativity, so

$$c = (c + a) + b.$$

Then c is the additive inverse of a so

$$c = 0 + b.$$

Finally, 0 is the additive identity so $c = b$, and the inverse is unique.

I leave the proofs for the multiplicative identities and inverses to the students. ■

This uniqueness of identities and inverses is important. It will used in some of the following proofs. Consider 0, the *additive* identity; what can be said of the *multiplication* $0 \cdot x$ for $x \in \mathbb{R}$? Similarly, -1 is the additive inverse of 1, so what can be said of the multiplication $(-1) \cdot x$? The only axiom that combines addition and multiplication is the distributive law and so it should be no surprise it is used within the proof of

Example B For all $x \in \mathbb{R}$, $0 \cdot x = 0$ and $(-1) \cdot x = -x$.

Solution Given $x \in \mathbb{R}$,

$$\begin{aligned} 0 \cdot x &= x \cdot 0 \quad \text{(A6)} \\ &= x \cdot (0 + 0) \quad \text{(A3)} \\ &= x \cdot 0 + x \cdot 0 \quad \text{(A9)} \\ &= 0 \cdot x + 0 \cdot x \quad \text{(A6)}. \end{aligned}$$

Add $-(0 \cdot x)$ to both sides.

$$\begin{aligned} 0 &= 0 \cdot x + (-(0 \cdot x)) \\ &= (0 \cdot x + 0 \cdot x) + (-(0 \cdot x)) \quad \text{(A1)} \\ &= 0 \cdot x + (0 \cdot x + (-(0 \cdot x))) \quad \text{(A4)} \\ &= 0 \cdot x + 0 \quad \text{(A3)} \\ &= 0 \cdot x. \end{aligned}$$

Next

$$\begin{aligned} x + (-1) \cdot x &= 1 \cdot x + (-1) \cdot x \quad \text{(A7)} \\ &= x \cdot 1 + x \cdot (-1) \quad \text{(A6)} \\ &= x \cdot (1 + -1) \quad \text{(A9)} \\ &= x \cdot 0 \quad \text{(A4)} \\ &= 0 \cdot x \quad \text{(A6)} \\ &= 0, \end{aligned}$$

by the first result of this example. This shows that $(-1) \cdot x$ is the additive inverse of x. Yet, by Example A, the additive inverse of an element is *unique*. Thus $(-1) \cdot x = -x$. ■

The second result here can be generalised, for all $x, y \in \mathbb{R}$,

$$(-x) \cdot y = -(x \cdot y) = x \cdot (-y).$$

What about inverses of inverses?

Example C For all $x \in \mathbb{R}$, $-(-x) = x$ and, provided $x \neq 0$, $(x^{-1})^{-1} = x$.

Solution Left to student. ■

Or sums and products of additive inverses?

Example D For all $x, y \in \mathbb{R}$, $(-x) + (-y) = -(x + y)$ and $(-x) \cdot (-y) = x \cdot y$.

Solution Left to student. ■

And a couple of often used results are

Theorem (**Cancellation Law for Addition**) If a, b and $c \in \mathbb{R}$ and $a + c = b + c$, then $a = b$.

Theorem (**Cancellation Law for Multiplication**) If a, b and $c \in \mathbb{R}, c \neq 0$ and $ac = bc$ then $a = b$.

Proofs Left to student. ■

For repeated multiplication we use the usual notation: for $a \in \mathbb{R}$, a^2 is an abbreviation for $a \cdot a$ (similarly a^3 is an abbreviation for $a \cdot (a \cdot a)$, etc.).

Example 1.1.1. Prove the identity $(x + y)^2 = x^2 + 2xy + y^2$ above.

Solution We have

$$\begin{aligned}
(x+y)^2 &= (x+y)(x+y) \quad \text{(by definition)}, \\
&= (x+y) \cdot x + (x+y) \cdot y \quad \text{(by A9)}, \\
&= x \cdot (x+y) + y \cdot (x+y) \quad \text{(by A6)}, \\
&= x \cdot x + x \cdot y + y \cdot x + y \cdot y \quad \text{(by A9)}, \\
&= x \cdot x + x \cdot y + x \cdot y + y \cdot y \quad \text{(by A6)}, \\
&= x \cdot x + 1 \cdot (x \cdot y) + 1 \cdot (x \cdot y) + y \cdot y \quad \text{(by A7)}, \\
&= x \cdot x + (1+1) \cdot (x \cdot y) + y \cdot y \quad \text{(by A9 and A6)}, \\
&= x \cdot x + 2 \cdot (x \cdot y) + y \cdot y \quad \text{(by definition)}, \\
&= x^2 + 2xy + y^2 \quad \text{(by definition)}.
\end{aligned}$$

Actually we have also used A1 many times here. This allowed us to ignore brackets in expressions involving many + symbols. ■

Finally, for now, we can define two further binary operations on $\mathbb{R}$:

Subtraction For all $x, y \in \mathbb{R}$, $x - y = x + (-y)$,

Division For all $x, y \in \mathbb{R}, y \neq 0$,

$$x \div y \ \left(= \frac{x}{y} = x/y\right) = x \cdot y^{-1}.$$

Example For all $s, t, x, y \in \mathbb{R}$ with $x, y \neq 0$ we have

- $(s-t)^2 = s^2 - 2st + t^2$,
- $(s-t)(s+t) = s^2 - t^2$,
- $(s/x)(t/y) = (st)/(xy)$,
- $(s/x) + (t/y) = (sy + tx)/(xy)$.

Solution Left to student. ■

Note that if we replace $\mathbb{R}$ in these axioms by $\mathbb{Q}$ then they still hold; that is, $\mathbb{Q}$ is also a field (but $\mathbb{Z}$ is not since it fails (A8)). The point of giving a name ("field") to an "arithmetic system" where these hold is that many more examples appear in mathematics, so it proved to be worth isolating these properties and investigating their general implications. Other fields that you will have encountered by now are the complex numbers and the integers modulo 5 (more generally, modulo any prime).

The real and rational numbers have some further properties not shared by, for example, the complex numbers or integers modulo 5.

Namely, the real numbers form an **ordered field**. This means that we have a **total order relation** $<$ on $\mathbb{R}$. All the properties of this relation and how it interacts with the arithmetic operations follow from the following axioms:

(Ord 1) $\forall a, b \in \mathbb{R}$, exactly one of the following is true: $a < b$ or $b < a$ or $a = b$; (*trichotomy*)

(Ord 2) $\forall a, b, c \in \mathbb{R}$, if $a < b$ and $b < c$, then $a < c$; (*transitivity*)

(Ord 3) $\forall a, b \in \mathbb{R}$, if $a < b$ then $\forall c \in \mathbb{R}$, $a + c < b + c$;

(Ord 4) $\forall a, b \in \mathbb{R}$, if $a < b$ then $\forall d \in \mathbb{R}^+$, $a \cdot d < b \cdot d$.

As usual, we write $a \leq b$ to mean either $a = b$ or $a < b$.

You do not need to remember exactly which rules have been written down here. The important point is just that this short list of properties is all that one needs in order to be able to prove all the other standard facts about $<$. For example, we have:

Example 1.1.2. Let us prove that if x is **positive** (i.e. $0 < x$) then $-x$ is **negative** (i.e. $-x < 0$). So suppose that $0 < x$. Then by (Ord 3), $0 + (-x) < x + (-x)$. Simplifying we get $-x = 0 + (-x) < x + (-x) = 0$, as required.

Other facts that follow from these properties include:

$\forall x \in \mathbb{R} \quad x^2 \geq 0$,

$\forall x, y \in \mathbb{R} \qquad x \leq y \Leftrightarrow -x \geq -y$, *et cetera.*

It left as an exercise to prove these facts using just the axioms (Ord 1–4). Also see the first exercise sheet. ■

However, there are more subtle facts about the real numbers that cannot be deduced from the axioms discussed so far. For example, consider the theorem that $\sqrt{2}$ is irrational. This really contains two statements: first that $\sqrt{2} \notin \mathbb{Q}$ (that is, no rational number squares to 2; this was proved in MATH10101/10111); second that there really is such a number in $\mathbb{R}$ - there is a (positive) solution to the equation $X^2 - 2 = 0$ in $\mathbb{R}$. And this definitely is an extra property of the real numbers—simply because it not true for the rational numbers (which satisfies all the algebraic and order axioms that we listed above).

So, we need to formulate a property of $\mathbb{R}$ that expresses that there are no "missing numbers" (like the "missing number" in $\mathbb{Q}$ where $\sqrt{2}$ should be). Of course, we have to say what "numbers" should be there, in order to make sense of saying that some of there are "missing". The example of $\sqrt{2}$ might suggest that we should have "enough" numbers so as to be able to take n-th roots of positive numbers and perhaps to solve other polynomial equations and following that idea does lead to another field - the field of real algebraic numbers - but we have a much stronger condition ("completeness") in mind here. We will introduce it in Chapter 2, just before we need it.

1.1.7 The Integer Part (or 'Floor') Function

From any of the standard constructions of the real numbers one has the fact that any real number is sandwiched between two successive integers in the following precise sense:

$\forall x \in \mathbb{R}$, $\exists n \in \mathbb{Z}$ *such that* $n \leq x < n + 1$.

(The proof of the existence of n requires the Completeness axiom, discussed later.) The integer n that appears here, the greatest integer less than or equal to x, is unique and is denoted $[x]$; it is called the **integer part** of x. The function $x \mapsto [x]$ is called **the integer part function**. Note that $0 \leq x - [x] < 1$.

Example 1.1.3. $[1.47] = 1$, $[\pi] = 3$, $[-1.47] = [-2]$.

Note Many people denote the *greatest integer less than or equal to* x by $\lfloor x \rfloor$, the **floor** function. In a similar manner, we can talk of the *least integer, greater than or equal to* x, denoted by $\lceil x \rceil$, the **ceiling function**.

We will often want to choose an integer larger than a given real number x. Instead of using the ceiling function we will choose $[x] + 1$; this is not always the least integer greater than or equal to x (consider when x is an integer) but it suffices for our needs.

Part I

Sequences

Chapter 2

Convergence

2.1 What is a Sequence?

Definition 2.1.1. *A* **sequence** *is a list $a_1, a_2, a_3, \ldots, a_n, \ldots$ of real numbers labelled (or* **indexed***) by natural numbers. We usually write such a sequence as $(a_n)_{n\in\mathbb{N}}$, or as $(a_n)_{n\geq 1}$ or just as (a_n). We say that a_n is the nth* **term of the sequence***.*

The word "sequence" suggests time, with the numbers occurring in temporal sequence: first a_1, then a_2, *et cetera*. Indeed, some sequences arise this way, for instance as successive approximations to some quantity we want to compute. Formally, a sequence is simply a function $f : \mathbb{N} \to \mathbb{R}$, where we write a_n for $f(n)$.

We shall be interested in the *long term behaviour* of sequences, i.e. the behaviour of the numbers a_n when n is very large; in particular, do the approximations converge on some value?

Example 2.1.2. Consider the sequence $1, 4, 9, 16, \ldots n^2, \ldots$. Here, the nth term is n^2. So we write the sequence as $(n^2)_{n\geq 1}$ or $(n^2)_{n\geq 1}$ or just (n^2). What is the nth term of the sequence $4, 9, 16, 25, \ldots$? What are the first few terms of the sequence $\big((n-1)^2\big)_{n\geq 1}$?

Example 2.1.3. Consider the sequence 2, 3/2, 4/3, 5/4, ... Here, $a_n = (n+1)/n$. The sequence is $((n+1)/n)_{n\geq 1}$.

Example 2.1.4. Consider the sequence $-1, 1, -1, 1, -1, \ldots$. A precise and succinct way of writing it is $\big((-1)^n\big)_{n\geq 1}$. The nth term is 1 if n is even and -1 if n is odd.

Example 2.1.5. Consider the sequence $((-1)^n/3^n)_{n\geq 1}$. The 5th term, for example, is $-1/243$.

Example 2.1.6. Sometimes we might not have a precise formula for the nth term but rather a rule for generating the sequence, E.g. consider the sequence $1, 1, 2, 3, 5, 8, 13, \ldots$, which is specified by the rule $a_1 = a_2 = 1$ and, for $n \geq 3$, $a_n = a_{n-1} + a_{n-2}$. (This is the **Fibonacci sequence**.)

Long term behaviour: In Examples 2.1.2 and 2.1.6 the terms get huge, with no bound on their size (we shall say that they **tend to** ∞).

However, for 2.1.3, the 100th term is

$$\frac{101}{100} = 1 + \frac{1}{100},$$

the 1000th term is

$$\frac{1001}{1000} = 1 + \frac{1}{1000}.$$

It looks as though the terms are getting closer and closer to 1. (Later we shall express this by saying that $((n+1)/n)_{n\geq 1}$ **converges** to 1.)

In Example 2.1.4 , the terms alternate between -1 and 1 so don't converge to a single value.

In Example 2.1.5, the terms alternate between being positive and negative, but they are also getting very small in *absolute* value (i.e. in their size when we ignore the minus sign): so $((-1)^n/3^n)_{n\geq 1}$ converges to 0.

Before giving the precise definition of "convergence" of a sequence, we require some technical properties of the modulus (i.e. absolute value) function.

We now make the convention that unless otherwise stated, all variables (x, y, l, ε, δ ...) range over the set $\mathbb{R}$ of real numbers.

2.2 The Triangle Inequality

We define the **modulus**, $|x|$ of x, also called the **absolute value** of x.

$$|x| = \begin{cases} x & \text{if } x \geq 0, \\ -x & \text{if } x < 0. \end{cases}$$

Note that $|x| = \max\{x, -x\}$, so $x \leq |x|$ and $-x \leq |x|$.

Theorem 2.2.1 (The Triangle Inequality). *For all x, y, we have*

$$|x + y| \leq |x| + |y|.$$

Proof. We have $x \leq |x|$ and $y \leq |y|$. Adding, we get $x + y \leq |x| + |y|$.
Also, $-x \leq |x|$ and $-y \leq |y|$, so $-(x + y) = (-x) + (-y) \leq |x| + |y|$.
It follows that $\max\{x + y, -(x + y)\} \leq |x| + |y|$, i.e. $|x + y| \leq |x| + |y|$, as required. ■

Remark: For $x, y \in \mathbb{R}$, $|x - y|$ is the distance from x to y along the "line" $\mathbb{R}$. The triangle inequality is saying that the sum of the lengths of any two sides of a triangle is at least as big as the length of the third side, as is made explicit in the following corollary. (Of course we are dealing here with rather degenerate triangles: the name really comes from the fact that in this form, the triangle inequality is also true for points in the plane.)

Corollary 2.2.2 (Also called the **Triangle Inequality**). *For all a, b, c, we have*

$$|a - c| \leq |a - b| + |b - c|.$$

Proof.

$$|a - c| = |(a - b) + (b - c)| \leq |a - b| + |b - c|$$

by Theorem 2.2.1 with $x = (a - b)$ and $y = (b - c)$). ■

Lemma 2.2.3. *Some further properties of the modulus function:*

(a) $\forall x, y \quad |x \cdot y| = |x| \cdot |y|$ *and, if* $y \neq 0$,

$$\left|\frac{x}{y}\right| = \frac{|x|}{|y|};$$

(b) $\forall x, y \quad |x - y| \geq \big||x| - |y|\big|$;

(c) $\forall x, l$ *and* $\forall \varepsilon > 0$, $\quad |x - l| < \varepsilon \iff l - \varepsilon < x < l + \varepsilon.$

Proof. Exercises: (a) - consider the cases; (b), (c) - see the Exercise Sheet for Week 3. ■

Now we come to the definition of what it means for a sequence $(a_n)_{n\geq1}$ to converge to a real number l. We want a precise mathematical way of saying that "as n gets bigger and bigger, a_n gets closer and closer to l" (in the sense the distance between a_n and l tends towards 0).

2.3 The Definition of Convergence

Definition 2.3.1. *We say that a sequence* $(a_n)_{n\geq 1}$ **converges to the real number** l *if the following holds:*

$$\forall \varepsilon > 0 \quad \exists N \in \mathbb{N} \textit{ such that for all } n \in \mathbb{N} \textit{ with } n \geq N \textit{ we have } |a_n - l| < \varepsilon.$$

Equivalently,

$$\forall \varepsilon > 0, \exists N \in \mathbb{N} : \forall n \geq 1, n \geq N \implies |a_n - l| < \varepsilon.$$

We use various expressions and notations for when this holds:

- a_n **tends to** l as n tends to infinity, written $a_n \to l$ as $n \to \infty$.
- the **limit** of $(a_n)_{n\geq 1}$ as n tends to infinity equals l, written $\lim_{n\to\infty} a_n = l$.

Example 2.3.2. Consider the sequence

$$\left(\frac{n+1}{n}\right)_{n\geq 1}$$

of 2.1.3.

Claim: $(n+1)/n \to 1$ as $n \to \infty$.

Proof Let $\varepsilon > 0$ be given. The definition requires us to show that there is a natural number N such that for all $n \geq N$,

$$\left|\frac{n+1}{n} - 1\right| < \varepsilon.$$

So we look for N so that $\forall n \geq N$, $|(1 + 1/n) - 1| < \varepsilon$, which is equivalent to requiring that $\forall n \geq N$, $|1/n| < \varepsilon$.

The N here will depend on ε (in fact in this example, as in most, no choice of N will work for all ε, so any choice of value for N will be in terms of ε). Since the last inequality is equivalent to $\forall n \geq N$, $1/\varepsilon < n$ we take N to be any natural number greater than $1/\varepsilon$, say for definiteness $[\varepsilon^{-1}] + 1$. ■

Note i. Any $N > 1/\varepsilon$ would have sufficed but, when you are required to show something exists, I would recommend you exhibit an explicit example, such as $[\varepsilon^{-1}] + 1$.

ii. The N depends *inversely* on ε; as ε gets smaller, the N gets larger. This is what we would expect, if we want to terms of the sequence to be closer to the limit value then we will have to look further along the sequence. Use this as a sanity check on your answers.

Example 2.3.3. Now consider the sequence

$$\left(\frac{(-1)^n}{3^n}\right)_{n\geq 1}$$

of 2.1.5. The *claim* is that $(-1)^n/3^n \to 0$ as $n \to \infty$.

Proof: Let $\varepsilon > 0$ be given. To prove the claim we must find $N \in \mathbb{N}$ so that $\forall n \geq N$,

$$\left|\frac{(-1)^n}{3^n} - 0\right| < \varepsilon.$$

In fact $N = [\varepsilon^{-1}] + 1$ (which implies $1/N < \varepsilon$) works here. For suppose that $n \geq N$. Then

$$\left|\frac{(-1)^n}{3^n} - 0\right| = \left|\frac{(-1)^n}{3^n}\right| = \frac{|(-1)^n|}{|3^n|}$$

by Lemma 2.2.3(a)

$$= \frac{1}{3^n} \leq \frac{1}{n}$$

since it is easy to show (by induction) that $\forall n \in \mathbb{N}$, $3^n \geq n$. But $1/n \leq 1/N < \varepsilon$, and we are done. ■

Note You could finish this example differently;

$$\left|\frac{(-1)^n}{3^n} - 0\right| = \frac{1}{3^n} \leq \frac{1}{3^N} < \varepsilon$$

as long as $3^N > 1/\varepsilon$, i.e. $N > \log_3(1/\varepsilon)$. This choose $N = [\log_3(1/\varepsilon)] + 1$. Though we will use the logarithm function in this course it will not be defined properly until next year, thus I recommend keeping away from it if at all possible. Also, I would claim $[\varepsilon^{-1}] + 1$ is 'simpler' than $[\log_3(1/\varepsilon)] + 1$ thus I would further recommend continuing and giving 'simple' upper bounds for $|a_n - l|$, e.g. here I consider $1/N$ to be simpler than $1/3^N$; that it is larger is immaterial, all we require is that the final bound can be made smaller than ε by taking N sufficiently large.

The definition of convergence is notoriously difficult for students to take in, and rather few grasp it straight away, especially to the extent of being able to apply it. So here's a different, but equivalent, way of saying the same thing. Maybe one of the definitions might make more sense to you than the other - it can be useful to look at something from different angles to understand it. And, of course, the more examples you do, the more quickly you will get the picture.

By Lemma 2.2.3(c) the condition $|a_n - l| < \varepsilon$ is equivalent to saying that $l - \varepsilon <$

$a_n < l + \varepsilon$. This in turn is equivalent to saying that $a_n \in (l - \varepsilon, l + \varepsilon)$. So, to say that $a_n \to l$ as $n \to \infty$ is saying that no matter how small an interval we take around l, the terms of the sequence $(a_n)_{n \geq 1}$ will eventually lie in it, meaning that, from some point on, *every* term lies in that interval.

You might like to look at the formula for N (in terms of ε) in Example 2.3.2 and check that (taking $\varepsilon = 1/10$),

$$\frac{n+1}{n} \in \left(1 - \frac{1}{10}, 1 + \frac{1}{10}\right)$$

for all $n \geq 11$, and that (taking $\varepsilon = 3/500$)

$$\frac{n+1}{n} \in \left(1 - \frac{3}{500}, 1 + \frac{3}{500}\right)$$

for all $n \geq 167$ (so, for $\varepsilon = 3/500$ we can take $N = 167$ or any integer ≥ 167 - the definition of convergence doesn't require us to choose the least N that will work).

Example 2.3.4. This is more of a non-example really. Consider the sequence $((-1)^n)_{n \geq 1}$ (of 2.1.4). Then there is no l such that the terms will eventually *all* lie in the interval $(l - \frac{1}{2}, l + \frac{1}{2})$. This is because if $l \leq \frac{1}{2}$ then the interval does not contain the number 1, yet infinitely many of the terms of the sequence are equal to 1, and if $l \geq \frac{1}{2}$ then the same argument applies to the number -1. Hence there is *no* number l such that $(-1)^n \to l$ as $n \to \infty$. ■

In general, if there is no l such that $a_n \to l$ as $n \to \infty$ then we say that the sequence $(a_n)_{n \geq 1}$ **does not converge** or **is divergent**.

Example 2.3.5. Another non-example. Consider the sequence $(n^2)_{n \geq 1}$ of Example 2.1.2. This does not converge either. Here is a rigorous proof of this fact directly using the definition of convergence.

Proof Suppose, for a contradiction, that there is some l such that $a_n \to l$ as $n \to \infty$. Choose $\varepsilon = 1$ in Definition 2.3.1. So there must exist some $N \in \mathbb{N}$ such that $|n^2 - l| < 1$ for all $n \geq N$. In particular $|N^2 - l| < 1$ and $|(N+1)^2 - l| < 1$. Therefore

$$|(N+1)^2 - N^2| = |(N+1)^2 - l + l - N^2| \leq |(N+1)^2 - l| + |l - N^2|,$$

by the Triangle Inequality. But each of the terms in the last expression here is less than 1, so we get that $|(N+1)^2 - N^2| < 1 + 1 = 2$. However, $|(N+1)^2 - N^2| = 2N + 1$, so $2N + 1 < 2$, which is absurd since $N \geq 1$. ■

This last example is a particular case of a general theorem. Namely, if the set of terms of a sequence is not bounded (as is certainly the case for the sequence $(n^2)_{n\geq 1}$) then it cannot converge. We develop this remark precisely now.

Definition 2.3.6. *Let S be any non-empty subset of $\mathbb{R}$.*

- *We say that S is* **bounded above** *if there is a real number M such that $\forall x \in S$, $x \leq M$. Such an M is an* **upper bound** *for S.*
- *Similarly, S is* **bounded below** *if there is a real number m such that $\forall x \in S$, $x \geq m$. Such an m is a* **lower bound** *for S.*
- *If S is both bounded above and bounded below, then S is* **bounded**.

Example 2.3.7. (1) Let $S = \{17, -6, -25, 25, 0\}$. Then S is bounded: an upper bound is 25 (or any larger number) and a lower bound is -25 (or any smaller number). In fact one can show easily by induction on the size of X that if X is a non-empty, finite subset of $\mathbb{R}$ then X is bounded.

(2) Intervals like $(a, b]$ or $[a, b]$, etc, for $a < b$ are bounded above, and below, by a, respectively b. However an interval like (a, ∞) is only bounded below.

(3) Let $S = \{x \in \mathbb{R} : x^2 < 2\}$. Since $1.5^2 > 2$ it follows that if $x^2 < 2$ then $-1.5 < x < 1.5$. Of course there are better bounds, but this is certainly sufficient to show that S is bounded.

Applying this to the set $\{a_n : n \in \mathbb{N}\}$ of terms of a sequence we make the following definition.

Definition 2.3.8. *A sequence $(a_n)_{n\geq 1}$ is* **bounded** *if there exists $M \in \mathbb{R}^+$ such that for all $n \in \mathbb{N}$, $|a_n| \leq M$.*

Theorem 2.3.9 (Convergent implies Bounded)**.** *Suppose that $(a_n)_{n\geq 1}$ is a convergent sequence (i.e. for some l, $a_n \to l$ as $n \to \infty$). Then $(a_n)_{n\geq 1}$ is a bounded sequence.*

Proof. Choose l so that $a_n \to l$ as $n \to \infty$. Now take $\varepsilon = 1$ in the definition of convergence. Then there is some $N \in \mathbb{N}$ such that for all $n \geq N$, $|a_n - l| < 1$.

But

$$|a_n| = |(a_n - l) + l| \leq |a_n - l| + |l|$$

by the triangle inequality. Thus for all $n \geq N$, $|a_n| \leq 1 + |l|$. So if we take

$$M = \max\{|a_1|, |a_2|, \ldots, |a_{N-1}|, 1 + |l|\}$$

we have that $|a_n| \leq M$ for *all* $n \in \mathbb{N}$, as required. ■

We give one last example before we prove some more general theorems about convergence.

Example 2.3.10. Let

$$a_n = \frac{8n^{1/3}}{n^3 + \sqrt{n}}.$$

Then $a_n \to 0$ as $n \to \infty$.

Proof: Let $\varepsilon > 0$ be given. We must find N (which will depend on the given ε) such that for all $n \geq N$,

$$\left| \frac{8n^{1/3}}{n^3 + \sqrt{n}} - 0 \right| < \varepsilon,$$

i.e. so that

$$\frac{8n^{1/3}}{n^3 + \sqrt{n}} < \varepsilon$$

since the terms are positive, we may remove the modulus signs.

So the game is to find a decent looking upper bound for $8n^{1/3}/(n^3 + \sqrt{n})$ so that one can easily read off what N must be. Well, we certainly have $n^3 \geq \sqrt{n}$ (for all n), so

$$\frac{8n^{1/3}}{n^3 + \sqrt{n}} \leq \frac{8n^{1/3}}{\sqrt{n} + \sqrt{n}} = \frac{4n^{1/3}}{\sqrt{n}}.$$

You must get used to tricks like this: replacing an expression in the denominator by a smaller expression and/or replacing the numerator by a larger term always increases the size of the fraction, *provided* everything is positive.

Now

$$\frac{4 \cdot n^{1/3}}{\sqrt{n}} = \frac{4 \cdot n^{1/3}}{n^{1/2}} = \frac{4}{n^{1/2-1/3}} = \frac{4}{n^{1/6}}.$$

So we have that

$$\frac{8n^{1/3}}{n^3 + \sqrt{n}} \leq \frac{4}{n^{1/6}}.$$

So we will have the desired inequality, namely

$$\left| \frac{8n^{1/3}}{n^3 + \sqrt{n}} - 0 \right| < \varepsilon,$$

provided that

$$\frac{4}{n^{1/6}} < \varepsilon$$

which, after rearrangement, is provided that $(4/\varepsilon)^6 < n$. So to complete the argument we just take N to be any natural number greater than $(4/\varepsilon)^6$, say $N = \left[(4/\varepsilon)^6\right] + 1$. ■

2.4 The Completeness Property for $\mathbb{R}$

Definition 2.4.1. *Let S be a non-empty subset of $\mathbb{R}$ and assume that S is bounded above. Then a real number M is a* **supremum** *or* **least upper bound** *of S if the following two conditions hold:*

(i) M is an upper bound for S, and

(ii) if $M' \in \mathbb{R}$ satisfies $M' < M$ then M' is not an upper bound for S.

It is easy to see that a set S cannot have more than one supremum. For if M_1 and M_2 were both suprema and $M_1 \neq M_2$, then either $M_1 < M_2$ or $M_2 < M_1$ (this is the sort of situation where one automatically uses rule (Ord 1)). But, in either case, this contradicts condition 2.4.1(ii) above.

Thus we have proved:

Lemma 2.4.2. *The supremum of a set S, if it exists, is unique and we then denote it by* $\sup(S)$. ■

Lemma 2.4.3. *Let S be a non-empty subset of $\mathbb{R}$ and assume that $M = \sup(S)$ exists. Then $\forall \varepsilon > 0$, $\exists x \in S$ such that $M - \varepsilon < x \leq M$.*

Proof. Let $\varepsilon > 0$ be given. Let $M' = M - \varepsilon$. Then $M' < M$, so by 2.4.1(ii), M' is *not* an upper bound for S. So there must be some $x \in S$ such that $x > M'$. Since M is an upper bound for S (by 2.4.1(i)), we also have $x \leq M$. So $M' < x \leq M$, i.e. $M - \varepsilon < x \leq M$, as required. ■

Example 2.4.4. For $S = \{17, -6, -25, 25, 0\}$ we clearly have that $\sup(S) = 25$. Indeed, if S is any non-empty, finite subset of $\mathbb{R}$ then it contains a greatest element and this element is necessarily its supremum.

Example 2.4.5. However, it need not be the case that a set contains its supremum as a member. Indeed, if $a < b$ then we claim that $\sup(a, b) = b$ despite the fact that $b \notin (a, b)$.

Proof. Let us prove this. Certainly b is an upper bound for (a, b). To see that 2.4.1(ii) is also satisfied, suppose that $M' < b$. Let $c = max\{a, M'\}$. Then certainly $c < b$ and so the *average*, $d = (c + b)/2$ of c and b satisfies $M' \leq c < d < b$. Similarly, $a < d < b$.

So d is an element of the set (a, b) which is strictly greater than M'. Hence M' is not an upper bound for (a, b), as required. ■

Does a supremum always exist?

Example *The set*

$$\mathcal{S}_{\mathbb{Q}} = \left\{x \in \mathbb{Q} : x^2 < 2\right\}$$

does not have a rational supremum.

Solution This is a non-empty set, bounded above, by 2 for example. Assume it has a least upper bound **in** $\mathbb{Q}$. Call this r. The rational numbers obey the axiom of trichotomy so either $r^2 = 2$, $r^2 < 2$ or $r^2 > 2$.

Case 1 It is a simple proof by contradiction that $r^2 = 2$ implies r is irrational. Contradicting the assumption that $r \in \mathbb{Q}$.

Case 2 If $r^2 < 2$ it is possible to find $\Delta_1 \in \mathbb{Q} : (r + \Delta_1)^2 < 2$. This means $r + \Delta_1 \in \mathcal{S}_{\mathbb{Q}}$, contradicting the assumption that r is an *upper bound* for $\mathcal{S}_{\mathbb{Q}}$.

To find Δ_1: with Δ_1 yet to be chosen consider

$$(r + \Delta_1)^2 = r^2 + 2r\Delta_1 + \Delta_1^2 = r^2 + \Delta_1 (2r + \Delta_1).$$

Demand $\Delta_1 < 1$ (strict), so $(r + \Delta_1)^2 < r^2 + \Delta_1 (2r + 1)$. Write $\Delta_1 = m/ (2r + 1)$, then $(r + \Delta_1)^2 < r^2 + m$. Finally choose $m = 2 - r^2$ to get $r^2 < 2$ as required.

To make this argument work we have to check that

$$\Delta_1 = \frac{2 - r^2}{2r + 1} < 1.$$

This holds, iff, $2 - r^2 < 2r + 1$ (and this requires $2r + 1 > 0$ to multiply up), which rearranges as $r^2 + 2r - 1 > 0$. Note that $1 \in \mathcal{S}_{\mathbb{Q}}$ and so $r \geq 1$, since r is, by definition, an upper bound for $\mathcal{S}_{\mathbb{Q}}$. And, for $r \geq 1$, we have $r^2 + 2r - 1 \geq 1 + 2 - 1 = 2 > 0$ as required.

Case 3 If $r^2 > 2$ it is possible to find $\Delta_2 \in \mathbb{Q} : (r - \Delta_2)^2 > 2$. If $x \in \mathcal{S}_{\mathbb{Q}}$ then

$$x^2 < 2 < (r - \Delta_2)^2,$$

i.e. $x < r - \Delta_2$. This means that $r - \Delta_2 \in \mathbb{Q}$ would be an upper bound for $\mathcal{S}_{\mathbb{Q}}$, contradicting the assumption that r is the *least* of all upper bounds for $\mathcal{S}_{\mathbb{Q}}$.

To find Δ_2: with Δ_2 yet to be chosen consider

$$(r-\Delta_2)^2 = r^2 - 2r\Delta_2 + \Delta_2^2 > r^2 - 2r\Delta_2.$$

Write $\Delta_2 = t/2r$ so $(r-\Delta_2)^2 > r^2 - t$. Finally, choose t to be half the distance from r^2 to 2, i.e. $t = (r^2-2)/2$. Then $\Delta_2 = (r^2-2)/4r \in \mathbb{Q}$ and

$$(r-\Delta_2)^2 > r^2 - \frac{r^2-2}{2} = \frac{r^2+2}{2} > 2$$

since $r^2 > 2$.

So in all three cases we are led to a contradiction. Thus our last assumption must be false, namely that $\mathcal{S}_{\mathbb{Q}}$ has a least upper bound in $\mathbb{Q}$. ■

Note that the supremum of $\mathcal{S}_{\mathbb{Q}}$ satisfies $x^2 = 2$. Just as we adding (or adjoined) the solution of $x^2 = -1$ to $\mathbb{R}$ in constructing the complex numbers $\mathbb{C}$, we could add in the positive solution of $x^2 = 2$, i.e. the supremum of $\mathcal{S}_{\mathbb{Q}}$, into $\mathbb{Q}$. We could continue, adding into the set of Rational numbers the least upper bounds of subsets of $\mathbb{Q}$. In reality, a real number is associated with a subset of rational numbers itself with no talk of supremum. Or rather associated with the set and it's compliment, a so-called Dedekind cut of the Rational numbers. A definition of Dedekind cuts can be found on p.12 of *Real Mathematical Analysis*, C.C. Pugh, Undergraduate Texts in Mathematics, Springer, 2017.

If you think of this construction of $\mathbb{R}$ as adding in the supremum of subsets of rational numbers what happens when we look at the supremum of subsets of *real* numbers? Will we have to add additional elements into $\mathbb{R}$ or will $\mathbb{R}$ suffice? The answer is that $\mathbb{R}$ is *complete*, we will not have to add in extra elements.

Property 2.4.6. *(The* **Completeness** *property of* $\mathbb{R}$*) (A14) Any non-empty subset of* $\mathbb{R}$ *which is bounded above has a supremum in* $\mathbb{R}$*.*

So we now have a difference between $\mathbb{R}$ and $\mathbb{Q}$; $\mathbb{R}$ is complete while $\mathbb{Q}$ is not complete.

Let us list some consequences of the completeness property.

Example 2.4.7. Consider now the set

$$S = \{x \in \mathbb{R} : x^2 < 2\}$$

of Example 2.3.7, (though note it differs from the $\mathcal{S}_{\mathbb{Q}}$ of the last example which contained only rational numbers). Then $s^2 = 2$ and so $s = \sqrt{2}$. (Clearly s is the positive square root of 2 since $1 \in S$, so $1 \le s$).

Solution The same ideas as in the solution of the previous example work here. By the Trichotomy of the Real Numbers we have one of $s^2 = 2$, $s^2 < 2$ or $s^2 > 2$.

If $s^2 < 2$ then, with $\Delta_1 = (2 - s^2)/(2s + 1)$, we have $(s + \Delta_1)^2 < 2$. This means that $s + \Delta_1 \in S$, contradicting the fact that s is an upper bound for S.

If $s^2 > 2$ then, with $\Delta_2 = (s^2 - 2)/4s$, we have $(s - \Delta_1)^2 > 2$. This means that $s - \Delta$ is an upper bound for S, contradicting the fact that s was the least of all upper bounds.

This means the third case must hold, i.e. $s^2 = 2$. ■

Note that we had to work harder in the solution to the Example on $\mathcal{S}_{\mathbb{Q}}$. In the earlier example we assumed the supremum, there called r, was rational, and we had to observe that the Δ_1 and Δ_2 were also rational so that we stayed within the Universe of Rational Numbers.

In fact, by using a similar method (though the details are rather more complicated), one can also show that

- for any $x \in \mathbb{R}$ with $x > 0$, and any natural number n, there exists $y \in \mathbb{R}$, with $y > 0$, such that $y^n = x$. Such a y is unique and is denoted $x^{1/n}$ or $\sqrt[n]{x}$ (the positive real nth root of x). Furthermore,
- for any $x \in \mathbb{R}$, and any *odd* natural number n, there exists a unique $y \in \mathbb{R}$ (of the same sign as x) such that $y^n = x$. Again, we write this y as $x^{1/n}$.

Finally, here is a property of the real numbers that you probably took for granted. However let's *prove* it using the completeness property.

Theorem 2.4.8. The Archimedean Property $\mathbb{N}$ *is not bounded above.*

Proof. To see this suppose, for a contradiction, that $\mathbb{N}$ is bounded above. Then the completeness property says that $\mathbb{N}$ has a supremum, say $s = \sup \mathbb{N}$. Apply Lemma 2.4.3 with $\varepsilon = 1/2$. Then the lemma guarantees that there is some $n \in \mathbb{N}$ such that $s - 1/2 < n$. Adding 1 to both sides we get $s + 1/2 < n + 1$ (Ord 3). But then we have $n + 1 \in \mathbb{N}$ and $s < n + 1$ which contradicts our assumption that s was the supremum of, and hence an upper bound for, S. ■

Corollary *For all* $\varepsilon > 0$ *there exists* $n \in \mathbb{N}$ *such that* $0 < 1/n < \varepsilon$.

Proof Assume for contradiction that there is no $n \in \mathbb{N}$ such that $0 < 1/n < \varepsilon$. Then for every $n \in \mathbb{N}$ it follows that $1/n \geq \varepsilon$ and hence $n \leq 1/\varepsilon$. This means that $1/\varepsilon$ is an upper bound for $\mathbb{N}$, contradicting the previous Theorem. Hence there is a $n \in \mathbb{N}$ such that $0 < 1/n < \varepsilon$. ■

One can use this result to prove:

Lemma 2.4.9. *$\forall x, y \in \mathbb{R}$, if $x < y$ then $\exists q \in \mathbb{Q}$ such that $x < q < y$.*

Proof Given $x, y \in \mathbb{R}$ with $x < y$ consider $y - x > 0$. By the Archimidean Property there exists $N \in \mathbb{N}$ such that $1/N < y - x$.

Claim There exists $R \in \mathbb{Z}$ with $x < R/N < y$.

Proof of claim Assume, for contradiction, that no such integer exists. By Completeness of $\mathbb{R}$,

$$\sup\left\{\frac{r}{N} \leq x : r \in \mathbb{Z}\right\}$$

exists. Completeness only says this supremum is a real number. In fact, it is of the form j/N for some $j \in \mathbb{Z}$. By definition of upper bound $(j+1)/N > x$ and so, by the assumption that there are no ration numbers between x and y, we must have $(j+1)/N \geq y$. This means that

$$\frac{j+1}{N} - \frac{j}{N} \geq y - x$$

i.e. $1/N \geq y - x$. This contradicts our choice of N. Thus the claim is proved and the main result follows. ■

Example *For all $x, y \in \mathbb{R}$, if $x < y$ then there exists an irrational γ such that $x < \gamma < y$.*

Solution Left to student. ■

We should also make the obvious corresponding definition for *lower* bounds for sets of real numbers.

Definition 2.4.10. *Suppose that S is a non-empty subset of $\mathbb{R}$ and that S is bounded below. Then a real number m is an* **infimum** *(or* **greatest lower bound***) of S if*

(i) m is a lower bound for S, and

(ii) if $m' \in \mathbb{R}$ and $m < m'$, then m' is not a lower bound for S.

Again, one can show that the infimum of a set S is unique if it exists, and we write $\inf(S)$ for it when it does exist. One might think that we need a result like Property 2.4.6 to postulate the existence of infima. But we can directly prove from that proposition.

Theorem 2.4.11 (Existence of infima)**.** *Every non-empty subset T of $\mathbb{R}$ which is bounded below has an infimum.*

Proof. See the Problem Sheet for Week 1. As a hint, define $T^- = \{-x : x \in T\}$. Then you should prove that T^- has a supremum, M say. Now show that $-M$ is the infimum of the original set T. ■

So, to summarise, we are not going to define the real numbers from first principles - you can find their construction in various sources and, in any case, you are used to dealing with them - but it is the case that everything we need about the real numbers follows from the axioms that we listed above for an ordered field, together with the completeness property. Indeed, these axioms completely characterise the real numbers. That is, and this is a rather remarkable fact: *although there are many different fields, and even many different ordered fields, if we add the completeness axiom then there is just one mathematical structure which satisfies all these conditions - namely the real numbers* $\mathbb{R}$.

2.5 Some General Theorems about Convergence

Theorem 2.5.1 (Uniqueness of Limits). *A sequence can converge to at most one limit.*

Proof. What we have to show is that if $a_n \to l_1$ as $n \to \infty$ and $a_n \to l_2$ as $n \to \infty$, then $l_1 = l_2$.

So suppose that $l_1 \neq l_2$. Say, wlog ("without loss of generality"), that $l_1 < l_2$. Let

$$\varepsilon = \frac{l_2 - l_1}{2},$$

so $\varepsilon > 0$.

Since $a_n \to l_1$ as $n \to \infty$, there exists $N_1 \geq 1$ such that for all $n \geq N$, $|a_n - l_1| < \varepsilon$.

Also, since $a_n \to l_2$ as $n \to \infty$, there exists $N_2 \geq 1$ such that for all $n \geq N'$, $|a_n - l_2| < \varepsilon$.

Now, let $m = \max\{N_1, N_2\}$. Then $|a_m - l_1| < \varepsilon$ and $|a_m - l_2| < \varepsilon$.

But

$$l_2 - l_1 = |l_2 - l_1| = |(l_2 - a_m) + (a_m - l_1)| \leq |l_2 - a_m| + |a_m - l_1|$$

by the triangle inequality. But the right hand side here is strictly less than $\varepsilon + \varepsilon$. That is, substituting in our value $\varepsilon = (l_2 - l_1)/2$, we obtain $l_2 - l_1 < l_2 - l_1$, a contradiction!

$\blacksquare$

So a sequence can have at most one limit. But when does it actually have a limit? The next piece of theory provides us with a criterion that covers many particular cases.

Definition 2.5.2. *A sequence $(a_n)_{n\geq 1}$ is said to be*

(i) **increasing** *if for all $n \in \mathbb{N}$, $a_n \leq a_{n+1}$;*

(ii) **decreasing** *if for all $n \in \mathbb{N}$, $a_{n+1} \leq a_n$;*

(iii) **strictly increasing** *if for all $n \in \mathbb{N}$, $a_n < a_{n+1}$;*

(iv) **strictly decreasing** *if for all $n \in \mathbb{N}$, $a_{n+1} < a_n$.*

A sequence satisfying any of these four conditions is called **monotone** *or* **monotonic**. *If it satisfies (iii) or (iv) it is called* **strictly monotonic**.

Theorem 2.5.3 (Monotone Convergence Theorem)**.** *Let $(a_n)_{n\geq 1}$ be a sequence that is both increasing* and *bounded. Then $(a_n)_{n\geq 1}$ converges.*

Proof. Since the set $\{a_n : n \in \mathbb{N}\}$ is bounded, it has a supremum, l say. We show that $a_n \to l$ as $n \to \infty$.

So let $\varepsilon > 0$ be given. We now use Lemma 2.4.3 which tells us that there is some $x \in \{a_n : n \in \mathbb{N}\}$ such that $l - \varepsilon < x \leq l$.

Obviously $x = a_N$ for some $N \in \mathbb{N}$. So $l - \varepsilon < a_N \leq l$.

Now for any $n \geq N$ we have that $a_n \geq a_N$ (since the sequence $(a_n)_{n\geq 1}$ is increasing) and of course we also have that $a_n \leq l$ (since l is certainly an upper bound for the set $\{a_n : n \in \mathbb{N}\}$, being its supremum).

So, for all $n \geq N$, we have $l - \varepsilon < a_n \leq l$, and hence $l - \varepsilon < a_n \leq l + \varepsilon$. But (by 2.2.3(c)) this is equivalent to saying that for all $n \geq N$, we have $|l - a_n| < \varepsilon$. Thus $a_n \to l$ as $n \to \infty$ as required. ■

Example 2.5.4. Let
$$a_n = \frac{n^2 - 1}{n^2}.$$
Then $(a_n)_{n\geq 1}$ is an increasing sequence since
$$a_n = 1 - \frac{1}{n^2} \leq 1 - \frac{1}{(n+1)^2} = a_{n+1}.$$

Further, $0 \leq a_n < 1$ for all n, so $(a_n)_{n\geq 1}$ is also a bounded sequence. Hence $(a_n)_{n\geq 1}$ converges.

In fact it is fairly easy to show directly from the definition of convergence that, with a_n as in the example above, $a_n \to 1$ as $n \to \infty$. However, Theorem 2.5.3 comes into its own in situations where it is far from easy to guess what the limit is:

Example 2.5.5. Let

$$a_n = \left(\frac{n+1}{n}\right)^n = \left(1 + \frac{1}{n}\right)^n.$$

Then one can show (though it's not particularly easy) that $(a_n)_{n\geq 1}$ is an increasing, bounded sequence. So it converges. But what is its limit? It turns out to be e (the base for natural logarithms).

2.6 Exponentiation - *a digression*

This subsection is really a digression, so you do not need to remember the details but it uses the ideas we have developed in an interesting way.

There are also some theoretical applications of Theorem 2.5.3 which allow one to give rigorous definitions of some classical functions of analysis. For example, if x and y are positive real numbers what does x^y, or "x to the power y", exactly mean? The answer "x multiplied by itself y times" doesn't really make sense if y is not a natural number! Here we see the idea of how to define this rigorously using Monotone Convergence although, since this is a bit of a digression, some steps will be skipped.

Let x, y be positive real numbers. We aim to give a rigorous definition of the real number x^y. If $y \in \mathbb{Q}$, say $y = n/m$ with $n, m \in \mathbb{N}$, then we may use the existence of roots (see Section 2.4) to define x^y to be $\sqrt[m]{x^n}$. But if $y \notin \mathbb{Q}$ how should we proceed? We shall use Question 5 of the Exercise sheet for Week 2, which tells us that we can approximate y arbitrarily closely by rational numbers q. We then show that the x^q converge and we call the limit x^y. The precise details are as follows.

Lemma 2.6.1. *(a) Let $y \in \mathbb{R}$. Then there exists a sequence $(q_n)_{n\geq 1}$ which (i) is strictly increasing, (ii) has rational terms (i.e. $q_n \in \mathbb{Q}$ for each $n \in \mathbb{N}$) and (iii) converges to y.*

(b) If q and s are positive rational numbers such that $q < s$, and if $x \geq 1$, then $x^q < x^s$.

Proof. (a) Firstly, let q_1 be any rational number strictly less than y (e.g. $q_1 = [y] - 1$.)

Let q_2 be any rational number satisfying

$$\max\left\{q_1, y - \frac{1}{2}\right\} < q_2 < y,$$

using the result from the problem sheet. Now let q_3 be any rational number satisfying

$$\max\left\{q_2, y - \frac{1}{3}\right\} < q_3 < y$$

as above.

We continue: once $q_1 < q_2 < \cdots < q_n < y$ have been constructed, we choose a rational number q_{n+1} satisfying

$$\max\left\{q_n, y - \frac{1}{n+1}\right\} < q_{n+1} < y.$$

Clearly our construction has ensured that the sequence $(q_n)_{n\geq 1}$ is (strictly) increasing and is bounded above (by y). It therefore converges by the Monotone Convergence Theorem. However, we have also guaranteed that for all $n \in \mathbb{N}$, $y - \frac{1}{n} < q_n < y$, from which it follows that the limit is, in fact, y.[1]

(b) This is left as an exercise; see the solution to Question 4 from the Exercise sheet for Week 3. ∎

Construction of x^y (outline).

We first use the lemma to construct an increasing sequence $(q_n)_{n\geq 1}$ of rational numbers that converges to y. Part (b) of the lemma implies that $(x^{q_n})_{n\geq 1}$ is a strictly increasing sequence which is bounded above by x^N where N is any natural number greater than y (e.g. $[y]+1$). Hence, by the Monotone Convergence Theorem, there is some l such that $x^{q_n} \to l \quad \text{as} \quad n \to \infty$. We define x^y to be this l.

We can extend this definition to negative y by the formula

$$x^{-y} = \frac{1}{x^y},$$

and we also set $x^0 = 1$. Finally, if $0 < x < 1$, so $x^{-1} \geq 1$, then we define x^y to be $(x^{-1})^{-y}$. We do not give any value to x^y for negative x. ∎

With these definitions all the usual laws for exponentiation can now be established. One first proves them for rational exponents and then shows that the laws carry over to arbitrary real exponents upon taking the limit described above. This latter process is made easier by developing an "algebra of limits" which we shall do in the next chapter.

The laws of exponentiation being referred to above are as follows (where x is assumed positive throughout).

[1] Can you see this final step? As a hint, note that for any $\varepsilon > 0$ there exists $n \in \mathbb{N}$ with $0 < \dfrac{1}{n+1} < \varepsilon$. Now use Lemma 2.2.3(c) as usual.

(E1) $x^{y+z} = x^y \cdot x^z$;

(E2) $(x \cdot y)^z = x^z \cdot y^z$ when $x, y > 0$;

(E3) $(x^y)^z = x^{y \cdot z}$;

(E4) if $0 < x < y$ and $0 < z$, then $x^z < y^z$.

Similarly, if $x \geq 1$ and $0 < z < t$, then $x^z < x^t$.

Since E1-E4 encapsulate all we will need to know about exponentiation, we don't need to refer back to our particular construction. In particular, we will use (E4) a number of times, without particular comment.

Chapter 3

The Calculation of Limits

We now develop a variety of methods and general results that will allow us to calculate limits of particular sequences without always having to go back to the original definition.

3.1 The Sandwich Rule

Roughly stated, this rule states that if a sequence is sandwiched between two sequences each of which converges to the same limit, then the sandwiched sequence converges to that limit as well. More precisely:

Theorem 3.1.1 (The Sandwich Rule)**.** *Let $(a_n)_{n\geq 1}$ and $(b_n)_{n\geq 1}$ be two sequences and suppose that they converge to the same real number l. Let $(c_n)_{n\geq 1}$ be another sequence such that for all sufficiently large n, $a_n \leq c_n \leq b_n$. Then $c_n \to l$ as $n \to \infty$.*

Proof. The hypotheses state that for some $N_0 \geq 1$ we have

$$a_n \leq c_n \leq b_n \qquad \text{for all } n \geq N.$$

Now we write down what it means for $\lim_{n\to\infty} a_n = \ell = \lim_{n\to\infty} b_n$. So, let $\varepsilon > 0$ be given.

Then there exists $N_1 \geq 1$ such that if $n \geq N_1$ then $|\ell - a_n| < \varepsilon$. Equivalently

$$\ell - \varepsilon < a_n < \ell + \varepsilon \qquad \text{for all } n \geq N_1.$$

Similarly, there exists $N_2 \geq 1$ such that if $n > N_2$ then

$$\ell - \varepsilon < b_n < \ell + \varepsilon \qquad \text{for all } n \geq N_2.$$

Now, let $M = \max\{N, N_1, N_2\}$. Then the displayed inequalities combine to show that:

$$\text{for all } n \geq M \text{ we have } \ell - \varepsilon < a_n \leq c_n \leq b_n < \ell + \varepsilon.$$

In particular, $\ell - \varepsilon < c_n < \ell + \varepsilon$, which is what we needed to prove to show that $\lim_{n\to\infty} c_n = \ell$. ■

Remark 3.1.2. The Sandwich Rule is often applied when either $(a_n)_{n\geq 1}$ or $(b_n)_{n\geq 1}$ is a *constant* sequence: if $a \leq c_n \leq b_n$ for sufficiently large n and $b_n \to a$ as $n \to \infty$, then $c_n \to a$ as $n \to \infty$. (Just take $a_n = a$ for all $n \in \mathbb{N}$ in 3.1.1.)

Similarly, if $a_n \leq c_n \leq b$ for sufficiently large n and $a_n \to b$ as $n \to \infty$, then $c_n \to b$ as $n \to \infty$.

Definition 3.1.3. *A sequence $(a_n)_{n\geq 1}$ is a* **null** *sequence if $a_n \to 0$ as $n \to \infty$.*

Theorem 3.1.4. *(i) $(a_n)_{n\geq 1}$ is a null sequence iff $(|a_n|)_{n\geq 1}$ is a null sequence.*

If $(a_n)_{n\geq 1}$ is a null sequence then $(-|a_n|)_{n\geq 1}$ is a null sequence.

(ii) (Sandwich rule for null sequences.) *Suppose that $(a_n)_{n\geq 1}$ is a null sequence. Let $(b_n)_{n\geq 1}$ be any sequence such that for all sufficiently large n, $|b_n| \leq |a_n|$. Then $(b_n)_{n\geq 1}$ is also a null sequence.*

Proof. (i)

$$|a_n - 0| = |a_n| = ||a_n|| = ||a_n| - 0|,$$

and so

$$|a_n - 0| < \varepsilon \quad \text{iff} \quad ||a_n| - 0| < \varepsilon,$$

which gives the required result.

The result on $(-|a_n|)_{n\geq 1}$ follows from $|a_n - 0| = ||a_n|| = |-|a_n| - 0|$.

(ii) We are given that for some $N \in \mathbb{N}$ we have $|b_n| \leq |a_n|$ for all $n \geq N$. It follows that for all $n \geq N$,

$$-|a_n| \leq b_n \leq |a_n|.$$

But by (i) both $(|a_n|)_{n\geq 1}$ and $(-|a_n|)_{n\geq 1}$ are null sequences. Hence by the Sandwich Rule 3.1.1 (with $l = 0$) it follows that $(b_n)_{n\geq 1}$ is null.

■

Of course, this is really what we were doing in explicit cases like Question 2 of the Week 3 Exercise Sheet.

Example 3.1.5. (i)
$$\left(\frac{1}{n}\right)_{n\geq 1}$$
is null.

(ii)
$$\left(\frac{1}{n^2+n^{3/2}}\right)_{n\geq 1}$$
is null.

Proof (i) We did this as part of Example 2.3.2. Choose $N = [1/\varepsilon]+1$ in the definition of limit.

(ii) For all $n \in \mathbb{N}$,
$$\left|\frac{1}{n^2+n^{3/2}}\right| = \frac{1}{n^2+n^{3/2}} \leq \frac{1}{n^2} = \left|\frac{1}{n^2}\right| \leq \left|\frac{1}{n}\right|.$$

So the result follows from Theorem 3.1.4 and Part (i) (Or you could use Question 2(a) of the Week 3 Exercise Sheet.) ■

Lemma 3.1.6. *(a) For all $m \geq 5$ we have $2^m > m^2$.*

(b) In particular, if $a_n = 2^{-n}$, then $\lim_{n\to\infty} a_n = 0$.

Proof. (a) You may have seen this in the *Foundations of Pure Mathematics* course.

Base case: For $m = 5$ we have $2^5 = 32 > 25 = 5^2$.

So, suppose for some $k \geq 5$ we have $2^k > k^2$. Then in trying to relate 2^{k+1} and $(k+1)^2$ one gets
$$2^{k+1} = 2\cdot 2^k \ > \ 2\cdot k^2 = (k+1)^2 + k^2 - 2k - 1.$$
Now the key point is that, as $k \geq 5$ we have $(k^2-2k-1) = (k-1)^2 - 2 > 2$. So from the displayed equation we get
$$2^{k+1} > (k+1)^2 + \big(k^2-2k-1\big) > (k+1)^2 + 2.$$

Thus, by induction $2^m > m^2$ for all $m \geq 5$.

(b) Of course, you can prove this more directly, but part (a) implies that $0 < a_n = 2^{-n} \leq n^{-2}$. Thus, by Theorem 3.1.4 and Question 2(a) (of Problem Sheet 2) $\lim_{n\to\infty}(a_n) = 0$.

■

Example 3.1.7.

$$\left(\frac{1}{2^n+3^n}\right)_{n\geq 1}$$

is null.

Proof: For all $n \in \mathbb{N}$, we have

$$\left|\frac{1}{2^n+3^n}\right| = \frac{1}{2^n+3^n} \leq \frac{1}{2^n}.$$

So the result follows from Theorem 3.1.4 and Lemma 3.1.6. ■

Example 3.1.8.

$$\left(\frac{n^3}{4^n}\right)_{n\geq 1}$$

is null.

Proof: By the lemma, $\forall n \geq 5$,

$$\left|\frac{n^3}{4^n}\right| = \frac{n^3}{4^n} = \frac{n^3}{2^n \cdot 2^n} \leq \frac{n^3}{n^2 \cdot n^2} = \frac{1}{n}.$$

So the result follows from 3.1.4 (and the fact that $(1/n)_{n\geq 1}$ is null). ■

3.2 The Algebra of Limits.

Hopefully you now have a feel for testing whether a given sequence is convergent or not. What should also be apparent is that we tend to repeat the same sorts of tricks. When that happens, one should suspect that there are general rules in play. This is the case here and we can convert the sorts of manipulations we have been doing into theorems telling us when certain types of functions converge (or not). The next theorem can be paraphrased as saying that: *Limits of convergent series satisfy the same rules of addition and multiplication as numbers.*

Theorem 3.2.1. *[The Algebra of Limits Theorem] Let* $(a_n)_{n\geq 1}$, $(b_n)_{n\geq 1}$ *be sequences and let* a, b *be real numbers. Suppose that* $\lim_{n\to\infty} a_n = a$ *and that* $\lim_{n\to\infty} b_n = b$. *Then:*

(i) $\lim_{n\to\infty} |a_n| = |a|$;

(ii) for any $k \in \mathbb{R}$, $\lim_{n\to\infty} ka_n = ka$;

(iii) $\lim_{n\to\infty}(a_n + b_n) = a + b$;

(iv) $\lim_{n\to\infty}(a_n \cdot b_n) = a \cdot b$;

(v) if $b \neq 0$, *then* $\lim_{n\to\infty} \dfrac{a_n}{b_n} = \dfrac{a}{b}$.

Important Remark: When we are assuming a limit exists, i.e. $\lim_{n\to\infty} a_n = a$, and we are given an $\varepsilon > 0$ along with a constant $C > 0$ we can feed $\varepsilon' = C\varepsilon$ into the definition of limit to find $M > 0 : m > M \implies |a_m - a| < C\varepsilon$.

Conversely, to prove that a limit exists, i.e. $\lim_{n\to\infty} a_n = a$, it suffices to prove that for all $\varepsilon > 0$ there exists $N > 0$ such that for all $n > N$ we have $|a_n - a| < \kappa\varepsilon$ for some constant $\kappa > 0$. I tend not to use this; when proving a limit exists I aspire to deduce $|a_n - a| < \varepsilon$, i.e. with $\kappa = 1$. This requires using the first half of this remark; applying any assumptions on limits with constant $C \neq 1$.

Proof. (i) Let $\varepsilon > 0$ be given. Choose N so that for all $n \geq N$, $|a_n - a| < \varepsilon$. Now by Lemma 2.2.3(b),

$$\big||a_n| - |a|\big| \leq |a_n - a|.$$

Hence, for all $n \geq N$, $\big||a_n| - |a|\big| < \varepsilon$, as required.

(ii) Firstly, if $k = 0$ the result is obvious since $\lim_{n\to\infty} 0 = 0$. So suppose that $k \neq 0$.

Let $\varepsilon > 0$ be given. Apply the Remark to obtain $M \in \mathbb{N}$ such that for all $m \geq M$, we have $|a_m - a| < \varepsilon/|k|$. Therefore, for $n \geq M$, we have

$$|ka_n - ka| = |k||(a_n - a)| < |k| \cdot \varepsilon/|k| = \varepsilon.$$

This shows that $ka_n \to ka$ as $n \to \infty$, as required.

(iii) Let $\varepsilon > 0$ be given. Apply the Remark to the two sequences. Thus we can find M_1 such that $|a_n - a| < \varepsilon/2$, i.e.

$$a - \varepsilon/2 < a_n < a + \varepsilon/2 \quad \text{for all } n \geq M_1$$

and M_2 such that $|b_n - b| < \varepsilon/2$, i.e.

$$b - \varepsilon/2 < b_n < b + \varepsilon/2 \quad \text{for all } n \geq M_2.$$

Adding these equations we find that

$$(a + b) - \varepsilon < (a_n + b_n) < (a + b) + \varepsilon$$

i.e. $|(a_n + b_n) - (a + b)| < \varepsilon$ for all $n \geq N = \max(M_1, M_2)$.

So $a_n + b_n \to a + b$ as $n \to \infty$, as required.

(iv) This is harder. Start by considering $|a_n b_n - ab|$ which we hope we can make small, i.e. $< \varepsilon$. But all we know is we can make $a_n - a$ and $b_n - b$ small, so we manipulate our problem to feed in what we know.

$$\begin{aligned} |a_n b_n - ab| &= |(a_n - a)\, b_n + ab_n - ab| \\ &= |(a_n - a)\, b_n + a\,(b_n - b)| \\ &\leq |(a_n - a)\, b_n| + |a\,(b_n - b)| \quad \text{by the triangle inequality} \\ &= |a_n - a|\,|b_n| + |a|\,|b_n - b| \quad \text{since } |xy| = |x|\,|y|\,. \end{aligned}$$

Now convergent sequences are bounded (Theorem 2.3.9) so there exists $B > 0 : |b_n| \leq B$ for all $n \geq 1$.

Let $\varepsilon > 0$ be given.

Using this ε in the definition of $\lim_{n\to\infty} a_n = a$ gives an $N_1 \geq 1$ such that

$$|a_n - a| < \frac{\varepsilon}{2B}.$$

Using the same ε is the definition of $\lim_{n\to\infty} b_n = b$ gives an $N_2 \geq 1$ such that

$$|b_n - b| < \frac{\varepsilon}{2\,(|a| + 1)}.$$

Note the $+1$ in the denominator; this is in case $a = 0$.

Let $N = \max(N_1, N_2)$. Assume $n \geq N$. For such n all the above results hold and combine as

$$\begin{aligned} |a_n b_n - ab| &\leq |a_n - a|\,|b_n| + |a|\,|b_n - b| \\ &< \frac{\varepsilon}{2B} B + |a| \frac{\varepsilon}{2\,(|a| + 1)} \\ &= \frac{\varepsilon}{2} + \frac{|a|}{(|a| + 1)} \frac{\varepsilon}{2} \\ &< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} \quad \text{since } |a|\,/(|a| + 1) < 1 \\ &= \varepsilon. \end{aligned}$$

Thus we have verified the definition that $\lim_{n\to\infty} a_n b_n = ab$.

(v) We start by proving that under the conditions in the Theorem,

$$\lim_{n\to\infty} \frac{1}{b_n} = \frac{1}{b}.$$

We are assuming $b \neq 0$ so $1/b$ is well-defined, but what of the $1/b_n$?

Claim: If $\lim_{n\to\infty} b_n = b$*,* $b_n \neq 0$ *(for all* n*) and* $b \neq 0$ *then there exists* $N_1 \in \mathbb{N}$ *such that* $\forall n \geq N_1$*, one has* $|b_n| > |b|/2 > 0$.

In particular, for such n, $b_n \neq 0$ and so the $1/b_n$ are well-defined.

Proof of Claim: Since $\lim_{n\to\infty} b_n = b$ we have, by Theorem 3.2.1, that $\lim_{n\to\infty} |b_n| = |b|$. Choose $\varepsilon = |b|/2 > 0$ in the definition of $\lim_{n\to\infty} |b_n| = |b|$ to find $N_1 \in \mathbb{N}$ such that $\forall n \geq N_1$, we have $||b_n| - |b|| < |b|/2$. Expand this as

$$|b| - \frac{1}{2}|b| < |b_n| < |b| + \frac{1}{2}|b|,$$

i.e.

$$\frac{1}{2}|b| < |b_n| < \frac{3}{2}|b|,$$

and the first inequality is what we wanted.

For $n \geq N_1$ we have

$$\left|\frac{1}{b_n} - \frac{1}{b}\right| = \left|\frac{b - b_n}{b_n b}\right| = \frac{|b - b_n|}{|b_n||b|}$$

However, $|b_n| > |b|/2$ (because $n \geq N_1$) and hence $|b_n| \cdot |b| > |b|^2/2$. Therefore

$$\frac{|b - b_n|}{|b_n||b|} < \frac{2}{|b|^2}|b - b_n|.$$

Let $\varepsilon > 0$ be given. By the Remark there exists $N_2 \in \mathbb{N}$ such that, for all $n \geq N_2$, we have

$$|b_n - b| < \frac{|b|^2\varepsilon}{2}.$$

Combine, so for $n \geq \max(N_1, N_2)$ we get

$$\left|\frac{1}{b_n} - \frac{1}{b}\right| < \frac{2}{|b|^2}|b - b_n| < \frac{2}{|b|^2}\frac{|b|^2\varepsilon}{2} = \varepsilon.$$

Thus we have verified the definition of

$$\lim_{n\to\infty} \frac{1}{b_n} = \frac{1}{b}.$$

Finally, for the general case

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} a_n \frac{1}{b_n} = \lim_{n\to\infty} a_n \lim_{n\to\infty} \frac{1}{b_n},$$

by the Product Rules for limits (part iv of this theorem), allowable since both limits on the right hand side exist. Thus

$$\lim_{n\to\infty} \frac{a_n}{b_n} = a\frac{1}{b} = \frac{a}{b}.$$

■

Theorem 3.2.2. *(i) Let $(a_n)_{n\geq 1}$ be a null sequence and let $(b_n)_{n\geq 1}$ be a bounded sequence (not necessarily convergent). Then $(a_n \cdot b_n)_{n\geq 1}$ is a null sequence.*

(ii) If $(|a_n|)_{n\geq 1}$ is a null sequence then $(a_n)_{n\geq 1}$ is null.

Proof. (i) Exercise.

(ii) Define the sequence $(b_n)_{n\geq 1}$ by $b_n = 1$ if $a_n \geq 0$ and $b_n = -1$ if $a_n < 0$. Then $a_n = |a_n| b_n$. Since the sequence $(b_n)_{n\geq 1}$ is bounded, we may apply the first part to the sequences $(|a_n|)_{n\geq 1}$ and $(b_n)_{n\geq 1}$, deducing that the sequence $(a_n)_{n\geq 1}$ is null if $(|a_n|)_{n\geq 1}$ is null. ■

Example 3.2.3. For any fixed positive real number p,

$$\frac{1}{n^p} \to 0$$

as $n \to \infty$.

Proof: Let $\varepsilon > 0$ be given. We want to show that $1/n^p < \varepsilon$ for all large n. But

$$\frac{1}{n^p} < \varepsilon \iff n^p > \frac{1}{\varepsilon} \iff n > \left(\frac{1}{\varepsilon}\right)^{1/p},$$

where the final equivalence uses E4 of Section 2.6. So, we take N to be $[\varepsilon^{-1/p}] + 1$. Thus, if $n \geq N$ then $n > \varepsilon^{-1/p}$ and so the above computation shows that $1/n^p < \varepsilon$. Therefore

$$\left|\frac{1}{n^p} - 0\right| < \varepsilon$$

and hence $\lim_{n\to 0} 1/n^p = 0$. ■

Example 3.2.4.

$$\frac{n^2 + n + 1}{n^2 - n + 1} \to 1$$

as $n \to \infty$

Proof: Divide top and bottom of the nth term by n^2. (This trick, and variations of it, is the main idea in all examples like this.) Thus

$$\frac{n^2 + n + 1}{n^2 - n + 1} = \frac{1 + \frac{1}{n} + \frac{1}{n^2}}{1 - \frac{1}{n} + \frac{1}{n^2}}.$$

We now apply the above example with $p = 1$ and $p = 2$ to deduce that $1/n \to 0$ and that $1/n^2 \to 0$ as $n \to \infty$. (Of course, these are also examples we have done several times before!)

So by the Algebra of Limits Theorem 3.2.1(ii, iii)

$$1 + \frac{1}{n} + \frac{1}{n^2} \longrightarrow 1 + 0 + 0 = 1 \qquad \text{as } n \to \infty$$

and

$$1 - \frac{1}{n} + \frac{1}{n^2} \longrightarrow 1 + (-1) \cdot 0 + 0 = 1 \qquad \text{as } n \to \infty.$$

So by using Algebra of Limits Theorem 3.2.1(vi)) again we obtain that

$$\frac{1 + \frac{1}{n} + \frac{1}{n^2}}{1 - \frac{1}{n} + \frac{1}{n^2}} \to \frac{1}{1} = 1,$$

that is,

$$\frac{n^2 + n + 1}{n^2 - n + 1} \to 1$$

as $n \to \infty$. ∎

For many functions given as polynomials or fractions of polynomials we can apply methods as in the above example. It is however very important to use this result (and earlier results) only for *convergent* sequences.

For example, we have seen that $(a_n)_{n\geq 1}$ is not convergent when $a_n = (-1)^n$. This is an example of a bounded sequence that is not convergent (so the converse of Theorem 2.5.3 fails).

There are however cases where we can deduce negative statements.

Example 3.2.5. If $p > 0$ then $(n^p)_{n\geq 1}$ is an unbounded sequence (and hence is not convergent by Theorem 2.3.9).

Proof. Given any $\ell > 0$ we can certainly find a natural number $n > (\ell)^{1/p}$ since the natural numbers are unbounded. However, using (E4) of Section 2.6 we have $n^p > \ell \iff n > (\ell)^{1/p}$. Thus this also proves that $(n^p)_{n\geq 1}$ is unbounded. ∎

It is worth emphasising that in proving unboundedness (or any counterexample) it's not necessary that every value of n is "bad", just that we can always find a larger "bad" value of n.

Example 3.2.6. (i) If $(a_n)_{n\geq 1}$ is unbounded and $(b_n)_{n\geq 1}$ is convergent, then $(a_n + b_n)_{n\geq 1}$ is unbounded. Similarly, $(ka_n)_{n\geq 1}$ is unbounded whenever $k \neq 0$.

Proof. Since (b_n) is convergent, it is bounded, say $|b_n| < B$ for all n. But now given any ℓ there exists some a_n with $|a_n| > B + \ell$. Hence by the triangle inequality, $|a_n + b_n| > \ell$, as required. The proof for products is similar. ∎

Chapter 4

Some Special Sequences

In the previous chapter we saw how to establish the convergence of certain sequences that were built out of ones that we already knew about. In this chapter we build up a stock of basic convergent sequences that will recur throughout your study of analysis.

4.1 Basic Sequences

Lemma 4.1.1. *For any $c > 0$, $c^{1/n} \to 1$ as $n \to \infty$.*

Proof. This proof has several steps, as follows:

Step I Prove that $(1+x)^n \geq 1 + nx$ for all $x \geq 0$ and $n \in \mathbb{N}$.

Step II By taking $x = y/n$ in Step I, deduce that for all $y > 0$ and $n \in \mathbb{N}$,

$$(1+y)^{1/n} \leq 1 + \frac{y}{n}.$$

Step III Hence show that for fixed $c > 1$, one has $c^{1/n} \to 1$ as $n \to \infty$.

Step IV Complete the proof.

Proofs of Steps:

Step I: Just use the binomial theorem—which says that

$$(1+x)^n = 1 + nx + (\text{lots of positive terms}).$$

Step II: Take n^{th} roots in Part I and use (E4) of Section 2.6 to get

$$1 + x \geq (1 + nx)^{1/n}.$$

Substituting $x = y/n$ gives

$$1 + \frac{y}{n} \geq (1 + y)^{1/n},$$

which is what we want.

Step III: Fix $\varepsilon > 0$. For any $c > 1$ we can write $c = 1 + y$ for $y > 0$. Now we choose $N = [y\varepsilon] + 1$, so that $y/N < \varepsilon$. Then for any $n \geq N$ we have:

$$\begin{aligned}
\left|c^{1/n} - 1\right| &= c^{1/n} - 1 && \text{since clearly } c^{1/n} > 1 \text{ (or use E4 of Section 2.6)} \\
&= (1+y)^{1/n} - 1 \\
&\leq \left(1 + \frac{y}{n}\right) - 1 && \text{by part II} \\
&= \frac{y}{n} \leq \frac{y}{N} < \varepsilon.
\end{aligned}$$

Step IV: So it remains to consider the case when $0 < c < 1$. But then $d = 1/c > 1$ so by Part III, $d^{1/n} \to 1$ as $n \to \infty$. By the Algebra of Limits Theorem 3.2.1(vi),

$$\frac{1}{d^{1/n}} \to \frac{1}{1} = 1$$

as $n \to \infty$. Finally, since

$$\frac{1}{d^{1/n}} = c^{1/n},$$

we have shown that $c^{1/n} \to 1$ as $n \to \infty$. ■

For completeness, recall Question 4 from the Week 4 Example Sheet:

Lemma 4.1.2. *For $0 < c < 1$ we have $c^n \to 0$ as $n \to \infty$.* □

Proof. Here is a different (and much simpler) proof. Let $d = 1/c > 1$ and write $d = 1+x$, so $x = 1/c - 1 > 0$. Then $d^n = (1+x)^n = 1 + nx + \cdots x^n$ by the binomial theorem. As all the terms are positive, $d^n > nx$. Thus for any number E, we have $d^n > E$ whenever $n \geq E/x$.

Assume $\varepsilon > 0$ is given. Then

$$c^n < \varepsilon \iff d^n = \frac{1}{c^n} > \frac{1}{\varepsilon}.$$

So, using the computations of the last paragraph, take $N = 1 + [1/x\varepsilon]$, where $x = 1/c - 1$. Then, for $n \geq N$ we have $n > 1/x\varepsilon$ and so $d^n > 1/\varepsilon$ by the last paragraph (with $E = 1/\varepsilon$). In other words, $c^n < \varepsilon$, as required. ∎

We now consider several sequences where it is harder to see what is going on; typically they are of the form $a_n = b_n/c_n$ where both b_n and c_n get large (or small) as n gets large. What matters, for the limit, is how quickly b_n grows (or shrinks to 0) in comparison with c_n. Roughly, if $(b_n)_{n\geq 1}$ has a higher "order of growth"[1] than c_n then the modulus of the ratio will tend to infinity and there will be no limit; if $(b_n)_{n\geq 1}$ and $(c_n)_{n\geq 1}$ have roughly the same order of growth, then we might get a limiting value for $(a_n)_{n\geq 1}$; if $(c_n)_{n\geq 1}$ has the higher order of growth, then $a_n \to 0$ as $n \to \infty$.

Lemma 4.1.3. *Suppose that $(a_n)_{n\geq 1}$ is a convergent sequence with limit ℓ. For any integer M, let $b_n = a_{n+M}$ (if M happens to be negative, we just take $b_n = 0$ or any other number for $1 \leq n \leq -M$). Then $\lim_{n\to\infty} b_n = \ell$.*

Proof. The point is that (apart from starting at different places) the two sequences are really the same, so ought to have the same limit. More formally, assume $\varepsilon > 0$ is given. Then we can find $N \in \mathbb{N}$ such that $|\ell - a_n| < \varepsilon$ for all $n \geq N$. Hence $|\ell - b_n| = |\ell - a_{n+M}| < \varepsilon$ for all for all $n \geq \max\{1, N - M\}$. So the sequence $(b_n)_{n\geq 1}$ converges. ∎

[1] By "order of growth" we mean something coarser than "rate of growth", that is, derivative. (The rate of growth = derivative is, however, relevant to computing limits, see L'Hôpital's Rule which we discuss later - it says that we can compute limits by replacing functions by their derivatives.

If $0 < c < 1$ we have seen that $c^n \to 0$ as $n \to \infty$ and so the next result is not surprising. But if $c \geq 1$ then $c^n \to +\infty$ as $n \to \infty$ and so the next result may not appear so obvious.

Lemma 4.1.4. *For any c,*

$$\frac{c^n}{n!} \to 0$$

as $n \to \infty$.

Proof. Set $a_n = c^n/n!$. Since it suffices to prove that $|a_n| \to 0$ (see Theorem 3.1.4), we can replace c by $|c|$ and assume that $c > 0$. In particular, $a_n > 0$ for all n.

Now,

$$a_{n+1} = a_n \cdot \frac{c}{n+1}.$$

For all $n \geq 2c$, we have

$$\frac{c}{n+1} < \frac{1}{2}$$

and hence

$$a_{n+1} = a_n \cdot \frac{c}{n+1} < \frac{a_n}{2}.$$

So, fix an integer $N \geq c$. Then for all $m > 0$ a simple induction says that

$$0 < a_{m+N} < a_N 2^{-m}.$$

Lemma 4.1.2 implies $\lim_{m \to 0} (2^{-m}) = 0$.

The Algebra of Limits, Theorem 3.2.1, implies $\lim_{m \to 0} a_N(2^{-m}) = 0$.

The Sandwich Theorem then gives $\lim_{m \to 0}(a_{m+N}) = 0$.

Finally, Lemma 4.1.3 gives $\lim_{n \to 0}(a_n) = 0$.

■

An interpretation of this is that when $c \geq 1$, $n!$ tends to infinity at a faster rate than c^n.

Next, two special sequences where the proofs are harder.

Lemma 4.1.5. *The sequence $(n^{1/n})_{n \geq 1}$ converges to 1 as $n \to \infty$.*

Proof. This is not so obvious.

Throughout the argument we consider only $n \geq 2$.

Let $k_n = n^{1/n} - 1$. Then E4 of Section 2.6 says that $k_n > 0$ and clearly $n = (1 + k_n)^n$.

By the binomial theorem

$$n \;=\; (1+k_n)^n \;=\; 1 + nk_n + \frac{n(n-1)}{2}k_n^2 + \cdots + k_n^n.$$

Since all the terms are positive, we can discard all but the third term, so

$$n > \frac{n(n-1)}{2}k_n^2,$$

and hence

$$k_n^2 < \frac{2n}{n(n-1)} = \frac{2}{n-1}.$$

So, for all $n \geq 2$ we have that

$$0 < k_n < \frac{\sqrt{2}}{\sqrt{n-1}}.$$

Now $\sqrt{2}/\sqrt{n-1} \to 0$ as $n \to \infty$ (exercise), so by the Sandwich Rule, $k_n \to 0$ as $n \to \infty$. But $n^{1/n} = 1 + k_n$, so by the Algebra of Limits Theorem, $n^{1/n} \to 1$ as $n \to \infty$.

■

We have seen that when $0 < c < 1$ and $r > 0$ we have $n^r \to +\infty$ and $c^n \to 0$ as $n \to \infty$. But what happens to the product $n^r c^n$, does it tend to $+\infty$, 0 or something in between?

Lemma 4.1.6. *Fix c with $0 < c < 1$ and fix r. Then $\lim_{n\to\infty}\ n^r \cdot c^n = 0$.*

Remark. This proof is quite subtle, and it is included for completeness. Once we have L'Hôpital's Theorem, we can give a very quick proof.

Proof. If $r = 0$ then the result is Lemma 4.1.5. Hence the result is also true for $r \leq 0$ by the Sandwich Rule (as $0 < n^r \cdot c^n \leq c^n$ if $r \leq 0$).

So we may assume that $r > 0$.

Let us first assume that $r \in \mathbb{N}$.

Let $x = (1/c) - 1$, then $x > 0$ since $0 < c < 1$. By the binomial theorem,

$$\left(\frac{1}{c}\right)^n = (1+x)^n = \sum_{i=0}^{n} \binom{n}{i} x^i.$$

Assume $n \geq 2r$. Then the sum includes the term $\binom{n}{r+1}x^{r+1}$ and all the others are positive, therefore

$$\left(\frac{1}{c}\right)^n > \binom{n}{r+1} x^{r+1} = \frac{n(n-1)\cdot \ldots \cdot (n-r)}{(r+1)!} x^{r+1}.$$

$$\frac{1}{n^r c^n} > n\left(\frac{n-1}{n}\right)\left(\frac{n-2}{n}\right)\cdots\left(\frac{n-r}{n}\right)\frac{x^{r+1}}{(k+1)!}.$$

Since $n \geq 2r$ we have, for $j \leq r$, $n \geq 2j$ and so $2n \geq 2j + n$, i.e. $2n - 2j \geq n$ and so

$$\frac{n-j}{n} \geq \frac{1}{2}.$$

Thus

$$\frac{1}{n^r c^n} > n\left(\frac{1}{2}\right)^k \frac{x^{r+1}}{(k+1)!} = An,$$

say, where $A = x^{r+1}/2^k(k+1)!$ is a constant not depending on n. Then

$$0 < n^r c^n < \frac{1}{An}.$$

Therefore $\lim_{n\to\infty} n^k c^n = 0$ by the Sandwich Theorem.

Now in the case that r is (positive and) not an integer we simply observe that $0 < n^r \cdot c^n \leq n^{[r]+1} \cdot c^n$ and, since $[r]+1$ *is* an integer we have that $n^{[r]+1} \cdot c^n \to 0$ as $n \to \infty$ by what we've just shown.

Hence $n^r \cdot c^n \to 0$ as $n \to \infty$ by the Sandwich Rule. ■

Example 4.1.7. Find:

(1) $\lim\limits_{n\to\infty} \sqrt[n]{3n}$.

(2) $\lim\limits_{n\to\infty} (-\frac{1}{2})^n$.

(3) $\lim\limits_{n\to\infty} \dfrac{n!}{n^n}$.

Proofs: (1) Here we can break it up to get $\sqrt[n]{3n} = 3^{1/n} n^{1/n}$. Now Lemmas 4.1.1 and 4.1.5 show that

$$\lim_{n\to\infty} 3^{1/n} = 1 = \lim_{n\to\infty} n^{1/n}.$$

Thus by the Algebra of Limits Theorem, $\lim_{n\to\infty} \sqrt[n]{3n} = 1$.

(2) We know from Lemma 4.1.2 that $\lim_{n\to\infty}(1/2)^n = 0$, and so Theorem 3.1.4 (or 3.2.2 implies that $\lim_{n\to\infty}(-1/2)^n = 0$.

(3) This does not follow directly from our results, but think about individual terms:

$$\frac{n!}{n^n} = \frac{1 \cdot 2 \cdot 3 \cdots n}{n \cdot n \cdot n \cdots n} = \frac{1}{n}\frac{2}{n} \cdots \frac{n}{n} \leq \frac{1}{n}.$$

Since all the terms are positive and $\lim_{n\to\infty} 1/n = 0$, we can use the Sandwich Theorem to get $\lim_{n\to\infty} n!/n^n = 0$. ■

4.2 New Sequences from Old

Relative orders of growth

The fastest growing function of n seen in our examples was n^n. From Example 4.1.7 (3) we find that

$$\frac{n!}{n^n} \to 0 \quad \text{as } n \to \infty.$$

Both $n!$ and n^n get arbitrarily large as n tends to infinity, but we read this result as saying that n^n has a *greater order of growth* than $n!$. Perhaps we could say that **the growth of n^n beats factorial growth**.

If $c > 1$ then c^n will grow arbitrarily large as n tends to infinity and we say it grows *exponentially quickly*. But Lemma 4.1.4 says that for all $c \in \mathbb{R}$,

$$\frac{c^n}{n!} \to 0 \quad \text{as } n \to \infty.$$

When $|c| > 1$ this shows that $n!$ grows quicker than exponentially. Perhaps we could say that **factorial growth beats exponential growth**.

If $r > 0$ then n^r will grow arbitrarily large as n tends to infinity and we say it grows *polynomially quickly*. Lemma 4.1.6 says that for all $r \in \mathbb{R}$ and all $0 < c < 1$,

$$n^r c^n \to 0 \quad \text{as } n \to \infty.$$

Rewriting this in terms of $d = 1/c > 1$ gives

$$\frac{n^r}{d^n} \to 0 \quad \text{as } n \to \infty.$$

This says that the growth of d^n, however close d is to 1, beats the growth of n^r, however large r is. Thus **exponential growth beats polynomial growth**.

We will not define the logarithm function, $\ln x$, in this course but it is useful to have it for examples. In MATH20101 it will be defined as the inverse of e^x. We will not define differentiation in this course but again it is a useful tool to have. In MATH20101 we will show that e^x is differentiable and, by the Inverse Function Theorem, $\ln x$ is then differentiable with

$$\frac{d}{dx} \ln x = \frac{1}{x},$$

for $x > 0$. Similarly, we do not define integration in this course but we will do so next year and prove the Fundamental Theorem of Calculus which justifies

$$\ln x = \int_1^x \frac{dt}{t},$$

for $x > 0$. Given these facts we now apply the idea that

$$\left| \int_a^b f(t)\, dt \right| \leq \int_a^b |f(t)|\, dt \leq \sup_{[a,b]} |f(t)|\, (b-a).$$

This gives

$$\ln x \leq \sup_{[1,x]} \frac{1}{t} (x-1) = (x-1) < x,$$

for $x \geq 1$. This can immediately be strengthened. Let $\varepsilon > 0$, so think of this as small. Then, this inequality with x replaced by x^ε gives $\ln x^\varepsilon < x^\varepsilon$, i.e.

$$\ln x < x^\varepsilon / \varepsilon.$$

This shows that the logarithm has an order of growth slower than polynomial, however small the power of the polynomial. That is, **polynomial growth beats logarithmic growth**.

For an explicit comparison of the logarithm with a power we have

Example 4.2.1. For any $c > 0$, we have

$$\lim_{n\to\infty} \frac{\ln n}{n^c} = 0.$$

Proof: Choose $\varepsilon = c/2$ above to get $\ln n < 2n^{c/2}/c$ for any integer $n \geq 1$. Thus

$$\frac{\ln n}{n^c} < \frac{2}{cn^{c/2}}.$$

The result then follows from the Sandwich Rule. ■

Finally we had results about functions that didn't grow arbitrarily large but converged to a finite limit. Lemma 4.1.4 gives

$$\forall c > 0,\ c^{1/n} \to 1 \quad \text{as } n \to \infty.$$

In fact we can replace the constant c by the increasing n as still get the same limit,

$$n^{1/n} \to 1 \quad \text{as } n \to \infty,$$

see Lemma 4.1.5.

We can list these results in a table.

Fastest-growing (top to bottom)	comments
n^n	See 4.1.7
$n!$	See 4.1.4
d^n for $1 < d$	See 4.1.6 and think of e^n
n^p for $p > 0$	See 3.2.3; you can also take any polynomial here
$\ln n$	see below
$c^{1/n}$ and $n^{1/n}$ for $c > 0$	These come last as they tend to 1; see 4.1.1 and 4.1.5.

Alternatively, we can say that a_n goes to zero faster than b_n if $\lim_{n\to\infty} a_n/b_n = 0$. We can invert the table, showing the functions ordered by how quickly they slip or plunge to zero.

Goes to zero fastest (top to bottom)	comments
$\dfrac{1}{n^n}$	See 4.1.7
$\dfrac{1}{n!}$	See 4.1.4
c^n for $0 < c < 1$	See 4.1.6 and think of $e^{-n} = \dfrac{1}{e^n}$
$\dfrac{1}{n^p} = n^{-p}$ for $p > 0$	you can put any polynomial in the denominator
$\dfrac{1}{\ln n}$	see below
$c^{1/n}$ and $n^{1/n}$ for $c > 0$	These come last as they tend to 1; see 4.1.1 and 4.1.5.

Example 4.2.2.

$$\frac{n^{100} + 2^n}{2^n + n^2} \to 1 \text{ as } n \to \infty.$$

The general method with sequences like this is to find the fastest growing term, and then divide top and bottom by it. In this example 2^n is the fastest-growing term.

Therefore we divide top and bottom by it:

$$\frac{n^{100}+2^n}{2^n+n^2} = \frac{\frac{n^{100}}{2^n}+1}{1+\frac{n^2}{2^n}}$$

Now $n^{100}/2^n = n^{100}\cdot(1/2)^n \to 0$ by Lemma 4.1.6 (with $k=100$ and $c=1/2$), and $n^2/2^n = n^2\cdot(1/2)^n \to 0$ by 4.1.6 (with $k=2$ and $c=1/2$).

Hence, by AoL,

$$\frac{n^{100}+2^n}{2^n+n^2} \to \frac{0+1}{1+0} = 1 \text{ as } n\to\infty.$$

■

Example 4.2.3.

$$\frac{10^{6n}+n!}{3\cdot n!-2^n} \to \frac{1}{3} \text{ as } n\to\infty.$$

The fastest-growing term is $n!$ so we divide top and bottom by it:

$$\frac{10^{6n}+n!}{3\cdot n!-2^n} = \frac{\frac{10^{6n}}{n!}+1}{3-\frac{2^n}{n!}} \to \frac{0+1}{3-0} = \frac{1}{3} \text{ as } n\to\infty.$$

Here we applied Lemma 4.1.4 with $c=10^6$ to deduce that $10^{6n}/n! \to 0$, and again with $c=2$ to deduce that $2^n/n! \to 0$, and finally we applied AoL. ■

Examples 4.2.4. Find the limits of the sequences (a_n) where:

$$(a)\ a_n = \frac{n^3}{3^n} \qquad (b)\ a_n = \frac{3^n}{n^3} \qquad (c)\ a_n = \frac{3^n+n!}{n!} \qquad (d)\ a_n = \frac{3!}{n^3} \qquad (e)\ \frac{n^{28}+5n^7+1}{2^n}.$$

Answers (a) By the table, exponentials grow faster than polynomials (or use 4.1.6), so this tends to zero.

(b) This is really something we deal with in the next chapter, but notice it is the reciprocal of (a). So, whereas in part (a) the terms of the sequence get arbitrarily small, in this part they will get arbitrarily large. We will say the sequence tends to infinity.

(c)

$$\frac{3^n+n!}{n!} = \frac{3^n}{n!}+1.$$

By the table, $\lim_{n\to\infty} 3^n/n! = 0$ and so by the AoL Theorem

$$\lim_{n\to\infty}\left(\frac{3^n}{n!}+1\right) = 1.$$

(d) Careful! This is just a constant times $1/n^3$ and so (by the AoL or directly) it tends to zero.

(e) Here the table says that the limit is 0. Note that to prove it using Lemma 4.1.6, you should write it as

$$\frac{n^{28}}{2^n} + 5\frac{n^7}{2^n} + \frac{1}{2^n}.$$

Then 4.1.6, says each fraction has limit zero and so the AoL says that

$$\lim_{n\to\infty}\frac{n^{28}+5n^7+1}{2^n} = 0 + 5\cdot 0 + 0 = 0.$$

■

In the next example we will use the following general result:

Lemma 4.2.5. *Suppose that $(a_n)_{n\geq 1}$ is a convergent sequence, say with $\lim_{n\to\infty} a_n = \ell$. Let r and s be real numbers such that $r \leq a_n \leq s$ for all sufficiently large $n \in \mathbb{N}$, say for all $n \geq M$. Then $r \leq \ell \leq s$.*

Proof. Suppose (for a contradiction) that, $\ell < r$. Let $\varepsilon = r - \ell$ and choose N_1 so that for all $n \geq N_1$, $|a_n - \ell| < \varepsilon$. Set $N = \max\{N_1, M\}$. Then for any $n \geq N$ we have

$$\ell - \varepsilon < a_n < \ell + \varepsilon = \ell + (r - \ell) = r.$$

This contradicts the hypothesis that $a_n \geq r$. The same argument shows that $\ell \leq s$. ■

Here is a more complicated type of example which we will see quite a lot.

Example 4.2.6. Let the sequence $(a_n)_{n\geq 1}$ be defined inductively by

$$a_1 = 2,$$

and for $n \geq 1$

$$a_{n+1} = \frac{a_n^2+2}{2a_n+1}.$$

We show that $(a_n)_{n\geq 1}$ converges and then find the limit.

To do this we first show that

(A) $\forall n \in \mathbb{N}$, $a_n \geq 1$, and

(B) $\forall n \in \mathbb{N}$, $a_{n+1} \leq a_n$.

Finally,

(C) This forces $(a_n)_{n\geq 1}$ to converge. Now use the AoL to solve an equation for the limit ℓ.

Proof of A: Now obviously $a_1 \geq 1$. So, suppose for induction, that for some natural number $k \geq 1$ we have $a_k \geq 1$. Then

$$a_{k+1} \geq 1 \iff \frac{a_k^2 + 2}{2a_k + 1} \geq 1 \iff a_k^2 + 2 \geq 2a_k + 1$$

(here we are using the fact that by induction $2a_k + 1$ is positive because a_k is). But

$$a_k^2 + 2 \geq 2a_k + 1 \iff a_k^2 - 2a_k + 1 \geq 0 \iff (a_k - 1)^2 \geq 0,$$

which is certainly true. So working back through these equivalences we see that $a_{k+1} \geq 1$ as required.

Remark. It is very important in this sort of argument that one has $\iff$ or at least $\Leftarrow$ since one is trying to prove the first statement using the validity of the final statement. If you just have $\Rightarrow$ you would not be justified in drawing these conclusions.

Proof of B: This is not by induction, but assume $n \geq 1$ is given. We have

$$a_{n+1} \leq a_n \iff \frac{a_n^2 + 2}{2a_n + 1} \leq a_n \iff a_n^2 + 2 \leq 2a_n^2 + a_n \iff 0 \leq a_n^2 + a_n - 2.$$

But, by part (A), $a_n \geq 1$ so $a_n^2 \geq 1$, and hence $a_n^2 + a_n \geq 2$, so we are done.

Step C. So we have now shown that $(a_n)_{n\geq 1}$ is a strictly decreasing sequence which is bounded (above by 2 and below by 1). Hence, by the Monotone Convergence Theorem (or, rather, by its variant in the Exercises for Week 3) it converges. Let its limit be ℓ. We calculate ℓ explicitly as follows.

We first note that, by Lemma 4.2.5, as $a_n \geq 0$ for all n, then $\ell \geq 0$ also holds.

By repeated use of the Algebra of Limits theorem, we have $a_n^2 \to \ell^2$, $a_n^2 + 2 \to \ell^2 + 2$, $2a_n + 1 \to 2\ell + 1$ and, finally,

$$\frac{a_n^2 + 2}{2a_n + 1} \to \frac{\ell^2 + 2}{2\ell + 1} \text{ as } n \to \infty$$

(note that the denominator is not zero since $\ell \geq 0$.) Thus

$$\lim_{n\to\infty} a_{n+1} = \frac{\ell^2+2}{2\ell+1}.$$

However, Lemma 4.1.3 shows that $\lim_{n\to\infty} a_{n+1} = \lim_{n\to\infty} a_n$. So putting these equations together we see that

$$\frac{\ell^2+2}{2\ell+1} = \ell.$$

We can now solve this for ℓ: rearranging the equation gives $\ell^2+2 = 2\ell^2+\ell$ and hence gives the quadratic equation $\ell^2+\ell-2=0$, or $(\ell+2)(\ell-1)=0$. Thus, either $\ell=1$ or $\ell=-2$. But we know that $\ell \geq 0$, so $\ell \neq -2$ and hence $\ell = 1$. Thus

$$\lim_{n\to\infty} a_n = 1.$$

Remarks 4.2.7. It is important to realise that *the calculation of the limit in Example 4.2.1 was valid only because we already knew that the limit* existed.

To bring this point home, reflect on the following (incorrect!) proof that $(-1)^n \to 0$ as $n \to \infty$ (which we know to be false from Example 2.1.4): Let $a_n = (-1)^n$. Then the sequence $(a_n)_{n\geq 1}$ is defined inductively by $a_1 = -1$ and $a_{n+1} = -a_n$. Let $l = \lim_{n\to\infty} a_n$. Then by the Algebra of Limits Theorem we have $\lim_{n\to\infty}(-a_n) = -l$, i.e. $\lim_{n\to\infty} a_{n+1} = -l$. But, as above, $\lim_{n\to\infty} a_n = \lim_{n\to\infty} a_{n+1}$, so $l = -l$, whence $2l = 0$ and so $l = 0$ as required!

More subtle examples will appear on future Exercise sheets.

4.3 Newton's Method for Finding Roots of Equations - *optional*

This sub-section is an aside that will not be covered in the class; as such, it is not examinable, but you might find it interesting and/or useful.

Suppose that we wish to find a solution to an equation of the form

$$f(x) = 0$$

where $f : \mathbb{R} \to \mathbb{R}$ is some function (e.g. we shall take $f(x)$ to be x^2-2 below and thereby look for a square root of 2).

Newton's method is as follows.

Let x_1 be a reasonable approximation to a solution of the equation. For $n \geq 1$, let

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}.$$

Then, under suitable assumptions on x_1 and f, it can be shown that the sequence $(x_n)_{n\geq 1}$ will converge to a solution and, further, is a very good way of finding closer and closer approximations to a solution.

Example 4.3.1. Let $x_1 = 1$ and $f(x) = x^2 - 2$. Then Newton's method suggests looking at the sequence defined inductively by

$$x_{n+1} = x_n - \left(\frac{x_n^2 - 2}{2x_n}\right)$$

Simplifying this gives:

$$x_{n+1} = \frac{x_n}{2} + \frac{1}{x_n}.$$

Now one can show that this sequences converges but let us just *assume* that here and let λ denote the limit. We check that $\lambda = \sqrt{2}$.

Now one can easily show that $\lambda \neq 0$, so we may apply the Algebra of Limits Theorem to obtain

$$\lim_{n\to\infty} \left(\frac{x_n}{2} + \frac{1}{x_n}\right) = \frac{\lambda}{2} + \frac{1}{\lambda}.$$

But also $x_{n+1} \to \lambda$ as $n \to \infty$, so $\lambda = \frac{\lambda}{2} + \frac{1}{\lambda}$ and it is a simple matter to solve this equation to obtain $\lambda = \sqrt{2}$ as required. ■

Let us now evaluate a few terms to see how well they approximate $\sqrt{2}$.

We have

$$\begin{aligned}
x_1 &= 1 \\
x_2 &= \frac{1}{2} + \frac{1}{1} = \frac{3}{2} \simeq 1 \cdot 5000000 \\
x_3 &= \frac{3}{4} + \frac{2}{3} = \frac{17}{12} \simeq 1 \cdot 4166667 \\
x_4 &= \frac{17}{24} + \frac{12}{17} = \frac{577}{408} \simeq 1 \cdot 4142156 \\
x_5 &= \frac{577}{816} + \frac{408}{577} \simeq 1 \cdot 4142136.
\end{aligned}$$

In fact, to seven decimal places we have that $\sqrt{2}$ is indeed equal to $1 \cdot 4142136$. ■

Example 4.3.2. Let us calculate $\sqrt{10}$ by Newton's method. We take $f(x) = x^2 - 10$ and $x_1 = 3$. The Newton sequence for this function is defined inductively by

$$x_{n+1} = x_n - \left(\frac{x_n^2 - 10}{2x_n} \right)$$

which, upon simplification, becomes

$$x_{n+1} = \frac{x_n}{2} + \frac{5}{x_n}.$$

[Check: If $x_n \to \lambda$ as $n \to \infty$, then by the usual argument, $\lambda = \frac{\lambda}{2} + \frac{5}{\lambda}$, which solves to $\lambda = \sqrt{10}$. So the limit, if it exists (which we assume here), is $\sqrt{10}$.]

We have

$$\begin{aligned}
x_1 &= 3 \\
x_2 &= \frac{3}{2} + \frac{5}{3} = \frac{19}{6} \simeq 3 \cdot 1666667 \\
x_3 &= \frac{19}{12} + \frac{30}{19} = \frac{721}{228} \simeq 3 \cdot 1622807
\end{aligned}$$

In fact, to seven decimal places, $\sqrt{10} = 3 \cdot 1622776$. ■

Chapter 5

Divergence

Example: Recall from Theorem 2.3.9 that if a sequence is convergent then it is bounded. We can turn this around and see that certain sequences are not convergent simply because they are not bounded; for example

$$\left(\frac{e^n}{n^2}\right)_{n\geq 1},$$

or some of the examples on the Exercise sheet for Week 6. To see this, let $K > 0$ be given. We know from Lemma 4.1.6 that $n^2/e^n \to 0$. Choose $\varepsilon = 1/K$ in the definition of limit to find N such that $n^2/e^n < 1/K$ for $n \geq N$. Taking reciprocals we get

$$\frac{e^n}{n^2} > K \qquad \text{for all } n \geq N.$$

So the sequence $(n^2/e^n)_{n\geq 1}$ is eventually larger than any given number, i.e. is unbounded and hence not convergent. Of course, this argument is completely general, so we will make a definition and a theorem out of it.

5.1 Sequences that Tend to Infinity

Definition 5.1.1. *A sequence $(a_n)_{n\geq 1}$ is* **divergent** *if it is not convergent, i.e. if there is no $l \in \mathbb{R}$ such that $a_n \to l$ as $n \to \infty$.*

Definition 5.1.2. *We say that a sequence $(a_n)_{n\geq 1}$* **tends to plus infinity as n tends to infinity** *if for each positive real number K, there exists an integer N (depending on K) such that for all $n \geq N$, $a_n > K$. That is*

$$\forall K > 0, \exists N > 0 : \forall n \geq 1, n \geq N \implies a_n > K.$$

In this case we write $a_n \to +\infty$ as $n \to \infty$, or $\lim_{n\to\infty} a_n = +\infty$.

Example 5.1.3. The sequence $((-1)^n)_{n\geq 1}$ is divergent since there is no $l \in \mathbb{R}$ such that $(-1)^n \to l$ as $n \to \infty$. (see Example 2.3.4). But it does not tend to infinity as $n \to \infty$ because one may simply take $K = 1$: there is clearly no N such that for all $n \geq N$, $(-1)^n > 1$.

Example 5.1.4. The sequence $(\sqrt{n})_{n\geq 1}$ is not bounded and so is divergent (by Theorem 2.3.9). It does tend to infinity as $n \to \infty$. For let $K > 0$ be given. Choose $N = [K^2] + 1$. Then if $n \geq N$ we have that $n \geq [K^2] + 1 > K^2$. Hence $\sqrt{n} > \sqrt{K^2} = K$, as required.

Example 5.1.5. Here is an example one needs to be careful about. *Consider the sequence $((-1)^n \cdot n)_{n\geq 1}$. This is clearly not a bounded sequence, so is not convergent. But it does not tend to infinity either.* For if it did, taking $K = 1$ we would have an N such that for all $n \geq \mathbb{N}$, $(-1)^n \cdot n > 1$. Which is absurd since half the time it is negative.

Theorem 5.1.6 (The Reciprocal Rule)**.** *(i) Let $(a_n)_{n\geq 1}$ be a sequence of non-zero real numbers such that $a_n \to +\infty$ as $n \to \infty$. Then*

$$\frac{1}{a_n} \to 0$$

as $n \to \infty$.

(ii) Let $(a_n)_{n\geq 1}$ be a sequence of non-zero real numbers such that for all sufficiently large n, $a_n > 0$. Assume that the sequence

$$\left(\frac{1}{a_n}\right)_{n\geq 1}$$

is null. Then $a_n \to +\infty$ as $n \to \infty$.

Proof. (i) Suppose that $a_n \to +\infty$ as $n \to \infty$ and let $\varepsilon > 0$ be given.

Taking $K = 1/\varepsilon > 0$ in Definition 5.1.2, there exists $N \in \mathbb{N}$ such that for all $n \geq N$, $a_n > K$, i.e. $a_n > 1/\varepsilon$. In particular these a_n are positive. Hence for all $n \geq N$,

$$\left|\frac{1}{a_n} - 0\right| = \frac{1}{a_n} < \frac{1}{1/\varepsilon} = \varepsilon,$$

which proves that $1/a_n \to 0$ as $n \to \infty$.

(ii) Now suppose that $1/a_n \to 0$ as $n \to \infty$. Let $K > 0$ be given.

Then $1/K > 0$, so taking $\varepsilon = 1/K$ we see that there exists an N such that for all

$n \geq N$,

$$\left| \frac{1}{a_n} - 0 \right| < \varepsilon.$$

We may also take N large enough so that we have $a_n > 0$ for all $n \geq N$. Thus for all $n \geq N$,

$$0 < \frac{1}{a_n} < \varepsilon = \frac{1}{K}.$$

Therefore for all $n \geq N$, $a_n > K$, so $\lim_{n\to\infty} a_n = +\infty$ as required. ■

Examples 5.1.7. One may invert the special null sequences of Section 4.1. For example, we have that for any $c > 0$, $n! \cdot c^n \to +\infty$ as $n \to \infty$. Similarly, if $c > 1$, we have $c^n/n^k \to +\infty$ as $n \to \infty$. (In both cases the proof is left as an exercise.)

Consider now the sequence $(n! - 8^n)_{n\geq 1}$. We have

$$\frac{1}{n! - 8^n} = \frac{1}{n!(1 - \frac{8^n}{n!})} = \frac{1}{n!} \cdot \frac{1}{(1 - \frac{8^n}{n!})} \to 0 \cdot \frac{1}{1-0} = 0,$$

by Lemma 4.1.4 with $c = 8$, and AoL. Also, the fact that $1 - 8^n/n! \to 1$ (as $n \to \infty$), ensures that for sufficiently large n, $n! - 8^n > 0$. Thus, by The Reciprocal Rule 5.1.6(ii), $n! - 8^n \to +\infty$ as $n \to \infty$.

We now prove an Algebra of Limits Theorem for sequences tending to infinity.

Theorem 5.1.8. *(i) Suppose that $(a_n)_{n\geq 1}$ and $(b_n)_{n\geq 1}$ both tend to plus infinity. Then*

(a) $a_n + b_n \to +\infty$ as $n \to \infty$;

(b) if $c > 0$, then $c \cdot a_n \to +\infty$ as $n \to \infty$;

(c) $a_n \cdot b_n \to +\infty$ as $n \to \infty$.

(ii) (The Infinite Sandwich Rule.) If $b_n \to +\infty$ as $n \to \infty$, and $(a_n)_{n\geq 1}$ is any sequence such that $a_n \geq b_n$ for all sufficiently large n, then $a_n \to +\infty$ as $n \to \infty$.

Proof. For (i)(b), let $K > 0$ be given. Then $K/c > 0$, so there exists $N \in \mathbb{N}$ such that $a_n > K/c$ for all $n \geq N$. Hence $c \cdot a_n > K$ for all $n \geq N$, and we are done.

The rest of the proofs are left as exercises. ■

Definition 5.1.9. *We say that a sequence $(a_n)_{n\geq 1}$* **tends to $-\infty$ as** *$n \to \infty$, if for all $K < 0$, there exists $N \in \mathbb{N}$ such that for all $n \geq N$, $a_n < K$. This is written: $a_n \to -\infty$ as $n \to \infty$.*

[This is easily seen to be equivalent to saying that $-a_n \to +\infty$ as $n \to \infty$.]

Example 5.1.10. The sequence $((-1)^n \cdot n)_{n\geq 1}$ is unbounded (so does not converge) but neither tends to $+\infty$ nor to $-\infty$ as $n \to \infty$.

Similarly $(8^n - n!) \to -\infty$ as $n \to \infty$. Why? Because this is exactly the same statement as saying that $-(8^n - n!) \to +\infty$ as $n \to \infty$. Which we proved in Examples 5.1.7.

Questions 6A:

Do the following sequences converge/diverge/tend to plus infinity or tend to minus infinity?

(a) $\left(\cos(n\pi)\sqrt{n}\right)_{n\geq 1}$

(b) $\left(\sin(n\pi)\sqrt{n}\right)_{n\geq 1}$

c) $\left(\dfrac{\sqrt{n^2+2}}{\sqrt{n}}\right)_{n\geq 1}$

(d) $\left(\dfrac{n^3+3^n}{n^2+2^n}\right)_{n\geq 1}$

e) $\left(\dfrac{n^2+2^n}{n^3+3^n}\right)_{n\geq 1}$

(f) $\left(\dfrac{1}{\sqrt{n}-\sqrt{2n}}\right)_{n\geq 1}$

Chapter 6

Subsequences

Looking at subsequences of a given sequence gives a practical test, Theorem 6.1.3, for non-convergence. These also feature in two of the most important ideas and results in analysis, namely Cauchy sequences and the Bolzano-Weierstrass theorem.

6.1 The Subsequence Test for Non-Convergence

Definition 6.1.1. *Suppose that $1 \leq k_1 < k_2 < \cdots < k_n < \cdots$ is a strictly increasing sequence of natural numbers. Then, given any sequence $(a_n)_{n \geq 1}$ of real numbers we can form the sequence $(a_{k_n})_{n \geq 1}$. Such a sequence is called a* **subsequence** *of the sequence $(a_n)_{n \geq 1}$. In other words, a subsequence of $(a_n)_{n \geq 1}$ is any sequence obtained by leaving out terms (as long as infinitely many are left in).*

Example 6.1.2. The sequence $(4^n)_{n \geq 1}$ is a subsequence of the sequence $(n^2)_{n \geq 1}$: we take $k_n = 2^n$. Then with $a_n = n^2$, we have $a_{k_n} = (k_n)^2 = (2^n)^2 = 2^{2n} = 4^n$.

Note As easy induction shows that for all $n \in \mathbb{N}$ we have $k_n \geq n$.

Theorem 6.1.3. *Let $(a_n)_{n \geq 1}$ be a sequence. For any subsequence $(a_{k_n})_{n \geq 1}$ of the sequence $(a_n)_{n \geq 1}$ we have:*

(i) if $a_n \to l$ as $n \to \infty$, then $a_{k_n} \to l$ as $n \to \infty$;

(ii) if $a_n \to +\infty$ as $n \to \infty$, then $a_{k_n} \to +\infty$ as $n \to \infty$;

(iii) if $a_n \to -\infty$ as $n \to \infty$ then $a_{k_n} \to -\infty$ as $n \to \infty$.

Proof. (i) Let $\varepsilon > 0$ be given. Choose $N \in \mathbb{N}$ so that for all $n \geq N$, $|a_n - l| < \varepsilon$. As noted above, for all $n \in \mathbb{N}$ we have $k_n \geq n$. Hence, if $n \geq N$ then $k_n \geq N$, so $|a_{k_n} - l| < \varepsilon$. So the same N works for the sequence $(a_{k_n})_{n \geq 1}$.

The proofs of (ii) and (iii) are similar. ■

As mentioned above, the practical importance of Theorem 6.1.3 is that it gives us a method of proving that certain sequences do **not** converge. We can either find two subsequences of the given sequence that converge to different limits, or else one subsequence that converges to either ∞ or $-\infty$.

Example 6.1.4. The sequence

$$\left((-1)^n + \frac{1}{n^2}\right)_{n\geq 1}$$

does not converge.

Proof Suppose it does, to l say. Consider the subsequence with $k_n = 2n$. This is the sequence

$$\left((-1)^{2n} + \frac{1}{(2n)^2}\right)_{n\geq 1} = \left(1 + \frac{1}{(2n)^2}\right)_{n\geq 1}$$

and this converges to 1. Hence, by Theorem 6.1.3(i), we must have $l = 1$. But now consider the subsequence with $k_n = 2n + 1$, i.e.

$$\left((-1)^{2n+1} + \frac{1}{(2n+1)^2}\right)_{n\geq 1} = \left(-1 + \frac{1}{(2n+1)^2}\right)_{n\geq 1},$$

and this converges to -1. Hence by 6.1.3(i) we must have $l = -1$, a contradiction. ■

Example 6.1.5. The sequence

$$\left(\frac{n}{4} - \left[\frac{n}{4}\right]\right)_{n\geq 1}$$

does not converge.

Proof Suppose it does, to l say. Consider the subsequence with $k_n = 4n$, i.e.

$$\left(\frac{4n}{4} - \left[\frac{4n}{4}\right]\right)_{n\geq 1}.$$

This is the sequence with all terms equal to 0 which of course converges to 0 and hence by 6.1.3(i), $l = 0$. However, consider now the subsequence with $k_n = 4n + 1$, i.e.

$$\left(\frac{4n+1}{4} - \left[\frac{4n+1}{4}\right]\right)_{n\geq 1}.$$

Note that

$$\left[\frac{4n+1}{4}\right] = \left[n + \frac{1}{4}\right] = n.$$

So this subsequence has all terms equal to $1/4$, and therefore it converges to $1/4$. So by 6.1.3(i), $l = 1/4$, a contradiction. ■

Example 6.1.6. The sequence

$$\left(n\sin\left(\frac{n\pi}{2}\right)\right)_{n\geq 1}$$

neither converges, nor tends to $+\infty$ nor tends to $-\infty$.

Proof First consider the subsequence with $k_n = 4n+1$, i.e.

$$\left((4n+1)\sin\left(\frac{(4n+1)\pi}{2}\right)\right)_{n\geq 1}.$$

Now

$$(4n+1)\sin\left(\frac{(4n+1)\pi}{2}\right) = (4n+1)\sin\left(2n\pi+\frac{\pi}{2}\right) = (4n+1)\sin\left(\frac{\pi}{2}\right) = 4n+1.$$

So this subsequence is $(4n+1)_{n\geq 1}$ which tends to $+\infty$ as $n\to\infty$.

Secondly, consider the subsequence with $k_n = 4n$, i.e. the sequence

$$\left(4n\sin\left(\frac{4n\pi}{2}\right)\right)_{n\geq 1}.$$

Every term here is 0 (since $\sin(2n\pi) = 0$ for all $n\in\mathbb{N}$). So this subsequence converges to 0. So by 6.1.3 (i), (ii) and (iii), the original sequence does not converge and nor does it tend to ∞ or $-\infty$. ■

Example 6.1.7. Does $\lim_{n\to\infty}\left(\sqrt{n}-[\sqrt{n}]\right)$ exist?

Hint: you may assume that $[\sqrt{m^2+m}] = m$.

Answer: Set $a_n = \sqrt{n}-[\sqrt{n}]$. We should use subsequences. One subsequence is pretty obvious—if we take $k_n = n^2$ then

$$a_{k_n} = \sqrt{n^2}-\left[\sqrt{n^2}\right] = n-n = 0,$$

so this subsequence clearly has limit 0.

For the second one we use the hint and take $k_n = n^2+n$; thus

$$a_{k_n} = \sqrt{n^2+n}-\left[\sqrt{n^2+n}\right] = \sqrt{n^2+n}-n.$$

This is now one of those cases where we use the trick

$$(x-y) = \frac{(x-y)(x+y)}{(x+y)}.$$

So, in this case

$$\begin{aligned}
\sqrt{n^2+n}-n = \quad & \frac{(\sqrt{n^2+n}-n)(\sqrt{n^2+n}+n)}{(\sqrt{n^2+n}+n)} = \frac{(n^2+n)-n^2)}{\sqrt{n^2+n}+n} \\
& = \frac{n}{\sqrt{n^2+n}+n} = \frac{1}{\sqrt{1+1/n}+1} \\
& \to \frac{1}{1+1} = \frac{1}{2}
\end{aligned}$$

as $n \to \infty$.

Since we have two subsequences of the original sequence with different limits the original sequence cannot have a limit. ■

(Here is the proof of the hint: Simply observe that

$$[\sqrt{m^2+m}] = m \iff \sqrt{m^2+m} < m+1 \iff m^2+m < (m+1)^2 = m^2+2m+1,$$

which is certainly true!)

6.2 Cauchy Sequences and the Bolzano-Weierstrass Theorem

We conclude this chapter with two fundamental results of analysis. You are not required to know their proofs in this course. However, since the proofs do not require any further knowledge on your part, and since they are such useful results, the detailed proofs are given in the Appendix to this chapter.

Theorem 6.2.1 (The Bolzano-Weierstrass Theorem (1817)). *Every bounded sequence (a_n) has a convergent subsequence.*

Remarks: (1) It is important to note that the Theorem is *not* saying that $(a_n)_{n\geq 1}$ is convergent—which is false in general. For instance take our favourite bad example, $a_n = (-1)^n$. So, for this example one would have to take a genuine subsequence. Two obvious examples would be $(a_2 = 1, a_4 = 1, a_6 = 1, ...)$ and $(a_1 = -1, a_3 = -1, a_5 = -1, ...)$

(2) The way to think about the theorem is as a generalisation of the Monotone Convergence Theorem—which says that if one has an *increasing* bounded sequence then it is convergent. So, you can probably guess how one might try to prove this theorem: given a bounded sequence $(a_n)_{n\geq 1}$ then start with a_1, and look for some $a_{k_2} \geq a_1$ and induct. If this is not possible then try to do the same with descending sequences. In fact the proof needs to be a bit more subtle, but it develops from this idea.

We now come to the problem of giving a necessary and sufficient condition on a sequence for it to converge, without actually knowing what the limit might be. The following definition is crucial and will feature in many different guises in your future Analysis and Topology courses.

Definition 6.2.2. *A sequence $(a_n)_{n\geq 1}$ is called a* **Cauchy sequence** *if for all $\varepsilon > 0$, there exists $N \in \mathbb{N}$ such that for all $i, j \in \mathbb{N}$ with $i, j \geq N$, we have $|a_i - a_j| < \varepsilon$.*

$$\forall \varepsilon > 0, \exists N \in \mathbb{N} : \forall i, j \in \mathbb{N}, i, j \geq N \implies |a_i - a_j| < \varepsilon.$$

So this is saying that the terms of the sequence $(a_n)_{n\geq 1}$ are getting closer and closer together. Then we have a central theorem.

Theorem 6.2.3. *A sequence converges if and only if it is a Cauchy sequence.*

It is important in the definition of a Cauchy sequence that we are saying that *all* the terms get close together. For example if one defined a sequence inductively by $a_1 = 1$ and

$a_n = a_{n-1} + 1/n$, then certainly for all $\varepsilon > 0$ we can find N such that $|a_n - a_{n+1}| < \varepsilon$ for $n \geq N$. But, in fact this sequence is not convergent (we have seen an informal proof of this already and will see it again when we consider infinite series). So, the last theorem would not be true if we just assumed that *successive* terms got close together.

6.2.1 Proofs for the section - *optional*

Here are detailed proofs of the Bolzano-Weierstrass Theorem and the resulting theorem on Cauchy sequences. As mentioned above, *these proofs are not required in this course.* But the results are so useful that, if you have the time do go through them, it will set you up nicely for when you study real and complex analysis and metric spaces next year.

Theorem 6.2.4. (The Bolzano-Weierstrass Theorem). *Any bounded sequence* (a_n) *possesses a convergent subsequence. In other words there exist a sequence of numbers*

$$k_1 < k_2 \cdots < k_r < k_{r+1} < \cdots$$

such that the subsequence $(a_{n_k})_{k\geq 1}$ *is convergent.*

Proof. Since $(a_n)_{n\geq 1}$ is bounded, it has a supremum $M = \sup\{a_n : n \in \mathbb{N}\}$ and an infimum $N = \inf\{a_n : n \in \mathbb{N}\}$. (see Chapter 2). The proof is quite sneaky; what we will do is construct a chain of real numbers

$$M_1 = M \geq M_2 \cdots \geq M_r \cdots \geq N$$

and a second sequence

$$N_1 = M_1 - 1,\ N_2 = M_2 - \frac{1}{2},\ N_3 = M_3 - \frac{1}{3},\ \cdots,\ N_r = M_r - \frac{1}{r} \cdots$$

such that some element a_{k_r} (with $k_r > k_{r-1}$) is squeezed between them: $N_r \leq a_{k_r} \leq M_r$. This will do. The reason is that by the Monotone Convergence Theorem, the $\{M_r\}$ have a limit, say μ. Then one shows that $\lim_{r\to\infty} N_r = \mu$ as well and hence by the Sandwich Theorem $\lim_{r\to\infty} a_{k_r} = \mu$ as well.

OK, let's see the details. As we said, we take $M_1 = M$ and $N_1 = M_1 - 1$. The point here is that N_1 is *not* an upper bound for $\{a_n : n \in \mathbb{N}\}$ and so there exists some a_{k_1} with $N_1 < a_{k_1} \leq M_1$. Now we let $M_2 = \sup\{a_n : n > k_1\}$.

We note here that any upper bound for the $\{a_n : n \in \mathbb{N}\}$ must also be an upper bound for the smaller set $\{a_n : n > k_1\}$. Thus $M_2 \leq M_1$.

This time we set $N_2 = M_2 - 1/2$. Then since N_2 is not an upper bound for $\{a_n : n > k_1\}$, there exists some a_{k_2} with $k_2 > k_1$ such that $N_2 < a_{k_2} \leq M_2$.

Now we induct. Suppose that we have found $(M_1, N_1, a_{k_1}, \cdots M_r, N_r, a_{k_r})$ in this way; they satisfy

$$N_i = M_i - \frac{1}{i} < a_{k_i} \leq M_i = \sup\{a_j : j > k_{i-1}\} \leq M_{i-1}$$

and $k_i > k_{i-1}$ for each $1 < i \leq r$.

Then we set $M_{r+1} = \sup\{a_j : j > k_r\}$ and $N_{r+1} = M_{r+1} - 1/(r+1)$. Then as N_{r+1} is not an upper bound of $\{a_j : j > k_r\}$ there exists some $a_{k_{r+1}}$ with $N_{r+1} < a_{k_{r+1}} \leq M_{r+1}$ and $k_{r+1} > k_r$. This completes the inductive step.

Now, the $\{M_\ell\}_{\ell \geq 1}$ is a descending sequence and, as each M_ℓ is an upper bound of some collection of the a_p, we see that the $M_\ell \geq a_p$ for some such a_p. Thus the M_ℓ are bounded below by N. Hence the Monotone Convergence Theorem from Q3 of the Examples for Week 3 implies that the limit $\lim_{r\to\infty} M_r$ exists, say $\lim_{r\to\infty} M_r = \mu$.

We next claim that $\lim_{r\to\infty} N_r = \mu$ as well. To see this pick $\varepsilon > 0$ and pick P such that if $p \geq P$ then

$$M_p - \mu = |M_p - \mu| < \varepsilon/2.$$

We also know that (for $q \geq Q = [2/\varepsilon] + 1$), we have

$$N_q \geq M_q - \frac{1}{q} \geq M_q - \frac{\varepsilon}{2}.$$

By the triangle inequality

$$|N_q - \mu| \leq |N_q - M_q| + |M_q - \mu| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon \qquad \text{for all } q \geq \max\{P, Q\}.$$

Thus, $\lim_{r\to\infty} N_r = \mu$, as claimed.

Now we are done: since $N_r < a_{k_r} \leq M_r$ for each r, the Sandwich Theorem ensures that $\lim_{r\to\infty} a_{k_r} = \mu$, as well.

Alternative proof Let $\{a_n\}_{n\geq 1}$ be a bounded sequence.

If the sequence only takes a finite number of distinct values then at least one of the values must be taken by infinitely many terms of the sequence. These terms form a convergent subsequence as required.

So we may assume the sequence takes an infinite number of distinct values, that is the *set* $\{a_n : n \geq 1\}$ is infinite.

Since $\{a_n\}_{n\geq 1}$ is a bounded sequence there exists a closed interval $I_0 \subseteq \mathbb{R} : a_n \in I_0$ for all $n \geq 1$.

We split I_0 at it's midway point into two closed sub-intervals. Since $\{a_n : n \geq 1\}$ is an infinite set the intersection of it with at least one of the two subintervals will be an infinite set. Call this subinterval I_1 (if both subintervals contain infinitely many elements of the set choose the left hand interval). Thus the set $\{a_n : a_n \in I_1\}$ is infinite. Choose k_1 to be one of the n occurring in the set (perhaps the minimum), so $a_{k_1} \in I_1$, and consider the set $\{a_n : n \in I_1, n > k_1\}$. This is still an infinite set since only a finite number of elements have been removed from the infinite set $\{a_n : a_n \in I_1\}$. Repeat the process.

Split I_1 at it's midway point into two closed sub-intervals. Choose the subinterval I_2 such that $\{a_n : n \in I_2, n > k_1\}$ is an infinite set. Choose $k_2 > k_1$ as the least n in this set, so $a_{k_2} \in I_2$, and consider the infinite set $\{a_n : n \in I_2, n > k_2\}$.

Continuing we have a sequence $k_1 < k_2 < k_3 < ...$ and of closed intervals $I_0 \supseteq I_1 \supseteq I_2 \supseteq I_3 \supseteq ...$ such that $a_{k_r} \in I_r$ for all $r \geq 1$. Further, if $\ell(I)$ is the length of the interval then $\ell(I_r) = 2^{-r}\ell(I_0)$.

These facts are sufficient to prove (i.e. it will not be done here) that $\bigcap_{r=0}^{\infty} I_r = \{a\}$ for some $a \in \mathbb{R}$ (it is important for this that the I_r are closed intervals), and $\lim_{r\to\infty} a_{k_r} = a$. Thus we have found a convergent subsequence. ■

Examples 6.2.5. The Theorem tells us nothing about what is the subsequence $\{a_{k_r}\}$ or its limit μ. Some exercises might give you a feel for the vagaries inherent in the argument.

For the sequence $a_n = 1 - 1/n$ for $n \geq 1$ all subsequences converge, and all to the same value, 1.

For a sequence like

$$a_n = (-1)^n \frac{1}{n}$$

there are lots of possible subsequences— you could take the descending chain $a_{2n} = 1/2n$ for all n or the ascending chain $a_{2n+1} = -1/(2n+1)$ for all n or some messier combination of the two.

The limit is also not uniquely determined; just think about the case of $a_n = (-1)^n$ for all n.

In fact one can extend the proof of this theorem a bit to prove the following:

Challenging Exercise: *Prove that every sequence* $(a_n)_{n\geq 1}$ *has a subsequence* $(a_{k_n})_{n\geq 1}$ *which is either increasing (i.e. for all* $n \geq 1, \quad a_{k_n} \leq a_{k_{n+1}}$*) or decreasing (i.e. for all* $n \geq 1, \quad a_{k_n} \geq a_{k_{n+1}}$*).*

We now consider Cauchy sequences as defined in Definition 6.2.2. We begin with:

Lemma 6.2.6. *Every Cauchy sequence $(a_n)_{n\geq 1}$ is bounded.*

Proof. Take N such that, for $i, j \geq N$ we have $|a_i - a_j| < 1$. In particular, taking $i = N$ we see that

$$a_N - 1 < a_j < a_N + 1 \qquad \text{for all } j \geq N.$$

Thus $a_n \leq \max\{a_1, a_2, \ldots, a_{N-1}, a_N + 1\}$ for all n. The lower bound is similar. ∎

We know that an increasing sequence $(a_n)_{n\geq 1}$ has a limit if and only if it is bounded (combine the Monotone Convergence Theorem with Theorem 2.3.9). Using Cauchy sequences we can get a general analogue:

Theorem 6.2.7. (The Cauchy Sequence Theorem) *A sequence $(a_n)_{n\geq 1}$ converges if and only if it is a Cauchy sequence.*

Proof. Suppose first that $(a_n)_{n\geq 1}$ is a Cauchy sequence. By the Bolzano-Weierstrass Theorem we can at least find a convergent subsequence; say $(a_{k_r})_{r\geq 1}$ with $\lim_{r\to\infty} a_{k_r} = \mu$.

Assume $\varepsilon > 0$ is given. By the definition of limit as $r \to \infty$ there exists $N_0 \geq 1$ such that

$$|a_{k_r} - \mu| < \frac{\varepsilon}{2} \qquad (1)$$

for all $r \geq N_0$.

Also, by the Cauchy condition there exists $N_1 \geq 1$ such that

$$|a_i - a_j| < \frac{\varepsilon}{2} \qquad (2)$$

for all $i, j \geq N_1$.

Set $N = \max\{N_0, N_1\}$. In (2) choose $i = k_r$ with $r > N$. Since $k_r \geq r$ we have $k_r > N \geq N_0$ and so (1) holds. By the triangle inequality these combine to give

$$|a_j - \mu| \leq |a_j - a_{k_r}| + |a_{k_r} - \mu| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon \qquad \text{for all } j \geq N.$$

Thus $\lim_{n\to\infty} a_n = \mu$ as we wanted.

Conversely, suppose that $(a_n)_{n\geq 1}$ converges; say with $\lim_{n\to\infty} a_n = \nu$. Let $\varepsilon > 0$ be given. There exists N such that $|a_j - \mu| < \varepsilon/2$ for all $j \geq N$. But now we find that for $i, j \geq N$ we have

$$|a_i - a_j| \;\leq\; |a_i - \nu] + |a_j - \nu] \;< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} \;= \varepsilon.$$

So we have a Cauchy sequence. ■

Chapter 7

L'Hôpital's Rule

We will use L'Hôpital's Rule, which some of you will have seen before. We can't mathematically justify the rule in this course unit, simply because it involves differentiation of functions and we have not yet given the rigorous definition of when a function is differentiable nor of what the derivative is. That will be done in the second year course on Real Analysis. But the rule is very useful, so we will use it.

We consider two sequences $(a_n)_{n\geq 1}$ and $(b_n)_{n\geq 1}$, both tending to infinity. L'Hôpital's Rule gives a method for calculating $\lim_{n\to\infty} a_n/b_n$ under certain circumstances.

7.1 L'Hôpital's Rule

Assume that $f : [1,\infty) \to \mathbb{R}$ and $g : [1,\infty) \to \mathbb{R}$ are two functions which can be differentiated at least twice and for some $N > 0$ that $g'(x) \neq 0$ for $x > N$. Set $a_n = f(n)$ and $b_n = g(n)$ for $n \in \mathbb{N}$.

$$\textbf{Assume that} \qquad \lim_{n\to\infty} a_n = +\infty = \lim_{n\to\infty} b_n. \qquad (\dagger)$$

Then

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{f'(n)}{g'(n)},$$

assuming that the Right Hand Side exists. Here $g' = dg/dx$.

Second Version: *L'Hôpital's Rule also holds if you replace* ($\dagger$) *by*

$$\textbf{Assume that} \qquad \lim_{n\to\infty} a_n = 0 = \lim_{n\to\infty} b_n. \qquad (\dagger\dagger)$$

Since we do not yet have a rigorous definition of the derivative, we're not in a position to have a rigorous proof of the validity of L'Hôpital's Rule, though you may have seen some arguments for it when studying Calculus. Anyway, we will take it as correct (which it is) and see some examples of its use.

Example 7.1.1. Say $a_n = 2n+3$ and $b_n = 3n+2$. Take $f(x) = 2x+3$ and $g(x) = 3x+2$, so that clearly

$$\frac{a_n}{b_n} = \frac{f(n)}{g(n)}$$

and the hypotheses of the rule are satisfied. Thus L'Hôpital's Rule tells us that

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{f'(n)}{g'(n)} = \lim_{n\to\infty} \frac{2}{3} = \frac{2}{3}.$$

Example 7.1.2. Let $a_n = 3n^2 - 4n + 2$ and $b_n = (n+1)^2$. Here we apply the rule to the functions $f(x) = 3x^2 - 4x + 2$ and $g(x) = (x+1)^2$, noting that they satisfy the conditions (twice-differentiability and the latter having nonzero derivative for large enough x). We obtain

$$\lim_{n\to\infty} \frac{3n^2 - 4n + 2}{(n+1)^2} = \lim_{n\to\infty} \frac{6n-4}{2(n+1)}.$$

Still top and bottom go to $+\infty$ as $n \to \infty$. But we may apply the rule again to obtain

$$\lim_{n\to\infty} \frac{6n-4}{2(n+1)} = \lim_{n\to\infty} \frac{6}{2} = 3.$$

Of course, in these examples one may also use the previously discussed method of dividing top and bottom by the fastest-growing term. However, L'Hôpital's Rule really comes into its own when applied to examples like the following. (They do all satisfy the differentiability condition and the condition, on the denominator, of being nonzero for large enough x, though we do not note this explicitly.)

Example 7.1.3. Consider the sequence

$$\left(\frac{\ln n}{n}\right)_{n\geq 1}.$$

Before applying L'Hôpital's Rule we should note that $\ln n \to \infty$ as $n \to +\infty$.

So we may indeed apply the Rule in this example (with $f(x) = \ln x$ and $g(x) = x$) and we see that

$$\lim_{n\to\infty} \frac{\ln n}{n} = \lim_{n\to\infty} \frac{\frac{1}{n}}{1} = \lim_{n\to\infty} \frac{1}{n} = 0.$$

Example 7.1.4. Similarly (with $f(x) = \ln x$ and $g(x) = \ln(x+1)$):

$$\lim_{n\to\infty} \frac{\ln n}{\ln(n+1)} = \lim_{n\to\infty} \frac{\frac{1}{n}}{\frac{1}{(n+1)}} = \lim_{n\to\infty} \frac{n+1}{n} = \lim_{n\to\infty} \left(1 + \frac{1}{n}\right) = 1.$$

Example 7.1.5. And a more complicated example:

$$\lim_{n\to\infty} \frac{\ln(4n^5-1)}{\ln(n^2+1)} = \lim_{n\to\infty} \left(\frac{20n^4}{4n^5-1} \cdot \frac{n^2+1}{2n}\right) = \lim_{n\to\infty} \frac{10n^3(n^2+1)}{4n^5-1}$$

$$= \lim_{n\to\infty} \frac{10n^5+10n^3}{4n^5-1} = \lim_{n\to\infty} \frac{10+\frac{10}{n^2}}{4-\frac{1}{n^5}} = \frac{10+0}{4-0} = \frac{5}{2}.$$

However, before applying L'Hôpital's Rule you have to be careful to ensure that (†) *holds, since otherwise you may get rubbish:*

Example 7.1.6. Take $a_n = 1 - \frac{1}{n}$ and $b_n = 2 - \frac{1}{n}$. then using the Rule we get

$$\lim_{n\to\infty} \frac{1-\frac{1}{n}}{2-\frac{1}{n}} = \lim_{n\to\infty} \frac{n^{-2}}{n^{-2}} = 1.$$

But even more obviously

$$\lim_{n\to\infty} \frac{1-\frac{1}{n}}{2-\frac{1}{n}} = \frac{1-0}{2-0} = \frac{1}{2}.$$

What went wrong? The problem is that (†) does not hold, since $\lim_{n\to\infty} a_n \neq \infty$. So, the rule should not have been applied.

Finally, L'Hôpital's Rule gives easy proofs of Lemma 4.1.6 and the related result for logs mentioned after the Tables on the Supplement to Section 5:

Example 7.1.7. For any $0 < c < 1$ and k we have $\lim_{n\to\infty} n^k c^n = 0$.

Proof: If $k \leq 0$ the result follows from $\lim_{n\to\infty} c^n = 0$. So we may assume $k > 0$.

The exponent k is a real number, not necessarily a natural number. So pick a natural number $K \geq k$. Since $n^K c^n \geq n^k c^n$, the Sandwich Rule says we need only prove that $\lim_{n\to\infty} n^K c^n = 0$. Secondly, we rewrite this as $\lim_{n\to\infty} n^K/d^n = 0$ (for $d = 1/c > 1$). This ensures that we are in a situation where we can apply L'Hôpital's Rule and gives

$$\lim_{n\to\infty} \frac{n^K}{d^n} = \lim_{n\to\infty} \frac{K}{\ln(d)} \frac{n^{K-1}}{d^n}.$$

By induction on K the RHS has limit 0 (where one starts the induction at $K-1=0$). $\square$

The rule for logs is similar: We are discussing $\lim_{n\to\infty} \ln n/n^c$, for any $c > 0$. Again we can apply the rule to get

$$\lim_{n\to\infty} \frac{\ln n}{n^c} \;=\; \lim_{n\to\infty} \frac{\frac{1}{n}}{cn^{c-1}} \;=\; \lim_{n\to\infty} \frac{1}{c}\,\frac{1}{n^c} \;\to\; 0.$$